MIXED TATE MOTIVES AND MULTIPLE ZETA VALUES.

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1. Introduction

Let l be a positive integer and $k_1, \dots, k_l$ be integers such that $k_i \geq 1$ for $i = 1, \dots, l-1$ and $k_l \geq 2$. We define the multiple zeta value $\zeta(k_1, \dots, k_l)$ of index $(k_1, \dots, k_l)$ as

$$\zeta(k_1, \dots, k_l) = \sum_{m_1 < \cdots < m_l} \frac{1}{m_1^{k_1} \cdots m_l^{k_l}}.$$

Set $h_0 = 0$ and $h_i = \sum_{j=1}^{i} k_i$. For an index $(k_1, \dots, k_l)$, the number $n = \sum_{i=1}^{l} k_i$ is called the weight of the index. The weight of the multiple zeta value $\zeta(k_1, \dots, k_l)$ is defined to be n. We have the following integral expression of $\zeta(k_1, \dots, k_l)$:

$$\zeta(k_1, \dots, k_l) = (-1)^l \int_{\Delta_n} \prod_{i=1}^{l} [\frac{dx_{h_{i-1}+1}}{x_{h_{i-1}+1} - 1} \wedge \prod_{j=1}^{k_i - 1} \frac{dx_{h_{i-1}+j+1}}{x_{h_{i-1}+j+1}}]$$

Here Δ_n is the topological cycle defined by $\{0 < x_1 < \cdots < x_n < 1\}$. We define the subspace L_n of $\mathbf{R}$ by

$$L_n = \sum_{\substack{k_1 + \cdots + k_l = n \\ k_i \geq 1, k_l \geq 2}} \zeta(k_1, \dots, k_l) \mathbf{Q}.$$

By the shuffle relation, we have $L_i \cdot L_j \subset L_{i+l}$ and the space $\mathcal{A} = \oplus_{n=0}^{\infty} L_n$ is a graded algebra. Zagier [1] conjectured the following dimension formula. (See also [2].) For the motivic interpretation, see [3].

Conjecture 1.1 (Zagier)**.**

$$\dim L_n = d_n,$$

where d_i is defined by the following inductive formula.

$$d_0 = 1, d_1 = 0, d_2 = 1, d_{i+3} = d_{i+1} + d_i \textit{ for } i \geq 0$$

The generating function of d_i is given by

$$\sum_{i=0}^{\infty} d_i t^i = \frac{1}{1 - t^2 - t^3}.$$

The main theorem of this paper is the following.

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Theorem 1.2 (Main Theorem). $\dim L_n \leq d_n$.

The same theorem is announced in [4] and [5]. Let us explain the outline of the proof. We construct a pair of varieties $(Y^0, \mathbf{B}^0)$ such that the periods of the relative cohomology $H^n = H^n(Y^0, j_! \mathbf{Q}_{Y^0 - \mathbf{B}^0})$ are expressed as $\mathbf{Q}$-linear combinations of multiple zeta values. Here j is the natural inclusion $Y^0 - \mathbf{B}^0 \hookrightarrow Y^0$. We show that the extension class of

$$
\begin{aligned}
0 \to W_{k+1}(H^n)/W_{k+2}(H^n) &\to W_k(H^n)/W_{k+2}(H^n) \\
&\to W_k(H^n)/W_{k+1}(H^n) \to 0
\end{aligned}
\tag{1.1}
$$

vanishes in the category of mixed Hodge structures for all k. We use the the following equality for the extension of mixed motives after Marc Levine [7].

$$
Hom_{DTM(\mathbf{Q})}(\mathbf{Q}, \mathbf{Q}(i)[1]) = K_{2i-1}(\mathbf{Q}) \otimes \mathbf{Q}
\tag{1.2}
$$

In particular, we have $Hom_{DTM(\mathbf{Q})}(\mathbf{Q}, \mathbf{Q}(1)[1]) = \mathbf{Q}^\times \otimes \mathbf{Q}$. By using the compatibility in Proposition 3.3, the extension class (1.1) is known to split in the category of mixed motives. Using this fact and equality (1.2), we prove that $H^n(Y^0, j_! \mathbf{Q}_{Y^0 - \mathbf{B}^0})$ is a subquotient of a direct sum of typical objects in the category of mixed motives.

Let us explain the contents of this paper. In Section 2, we construct by a succession of blowing ups a variety Y^0 and its closed subvariety $\mathbf{B}^0$ associated to multiple zeta values. We compute the relative de Rham cohomology $H^i(j_! K_{Y^0 - \mathbf{B}^0, DR})$. Since $H^i(j_! K_{Y^0 - \mathbf{B}^0, DR}) = 0$ for $i \neq n$, the cone $Cone(\mathbf{Q}_{Y^0} \to \mathbf{Q}_{\mathbf{B}^0})[n]$ defines an abelian object $\mathcal{A}_{TM}$ in $DTM_{\mathbf{Q}}$. In Section 3, we recall the definition of the abelian category of mixed Tate motives and prove several propositions concerning generators of abelian sub-categories in $\mathcal{A}_{TM}$. We also prove the compatibility of the cycle map for mixed Tate motives and the map ch for the extensions of mixed Tate Hodge structures. In Section 4, we compute the periods of the relative cohomologies defined in Section 2. We claim that the periods of the relative cohomologies are expressed by multiple zeta values. In the last section, we prove the Main Theorem.

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2. Varieties associated to multiple zeta values

2.1. **Successive blowing up of affine spaces.** In this section, we introduce a successive blowing up of $(\mathbf{A}^1)^n$.

Let $(\mathbf{A}^1)^n = \{(x_1, \dots, x_n)\}$ be a product of $\mathbf{A}^1$. We define a sequence of subvarieties Z_i, W_i for $i = 0, \dots, n-2$ by

$$
\begin{aligned}
Z_i &= \{(x_1, \dots, x_n) \mid x_1 = \cdots = x_{n-i} = 0\}, \\
W_i &= \{(x_1, \dots, x_n) \mid x_{i+1} = \cdots = x_n = 1\},
\end{aligned}
$$

We define sequences of blowing ups and their centers

$$\begin{array}{ccccccccc}
(\mathbf{A}^1)^n = X_0 & \leftarrow & X_1 & \leftarrow & \cdots & \leftarrow & X_{n-2} & \leftarrow & X_{n-1} = X \\
\cup & & \cup & & & & \cup & & \\
Z_0^{pr} & & Z_1^{pr} & & & & Z_{n-2}^{pr} & &
\end{array}$$

$$\begin{array}{ccccccccc}
X = Y_0 & \leftarrow & Y_1 & \leftarrow & \cdots & \leftarrow & Y_{n-2} & \leftarrow & Y_{n-1} = Y \\
\cup & & \cup & & & & \cup & & \\
W_0^{pr} & & W_1^{pr} & & & & W_{n-2}^{pr} & &
\end{array}$$

by the following procedure.

1. X_0 (resp. Y_0) is equal to $(\mathbf{A}^1)^n$ (resp. X).
2. Let Z_i^{pr} (W_i^{pr}) be the proper transform of Z_i (resp. W_i) under the morphism $X_i \to X_0$ (resp, $Y_i \to X_0$).
3. Let X_{i+1} (resp. Y_{i+1}) be the blowing up of X_i (resp. Y_i) with the center Z_i^{pr} (resp. W_i^{pr}) for $i = 1, \dots, n-2$.

We define divisors D_i^ϵ ($\epsilon = 0, 1, 1 \le i \le n$) of $(\mathbf{A}^1)^n$ by

$$D_i^\epsilon = \{x_i = \epsilon\}$$

and

$$D = D_2^0 \cup \cdots \cup D_n^0 \cup D_1^1 \cup \cdots \cup D_{n-1}^1$$

and the proper transform of D under the morphism $X_i \to X_0$ (resp. $Y_i \to X_0$, $X \to X_0$, $Y \to X_0$) is denoted by D_{X_i} (resp. D_{Y_i}, D_X, D_Y). The proper transform of D_j^ϵ under the morphism $X_i \to X_0$ (resp. $Y_i \to X_0$, $X \to X_0$, $Y \to X_0$) is denoted by D_{j,X_i}^ϵ (resp. D_{j,Y_i}^ϵ, $D_{j,X}^\epsilon$, $D_{j,Y}^\epsilon$). The open subvariety $X_i - D_{X_i}$ (resp. $Y_i - D_{Y_i}$, $X - D_X$, $Y - D_Y$) is denoted by X_i^0 (resp. Y_i^0, X^0, Y^0).

We define Z_i^0 and W_i^0 by $Z_i^{pr} \cap X_i^0$ and $W_i^{pr} \cap Y_i^0$, respectively. It is easy to see that the divisors $D_{1,X_i}^0, \dots, D_{n-i,X_i}^0$ (resp. $D_{i+1,Y_i}^1, \dots, D_{n,Y_i}^1$) are normal crossing and $Z_i^{pr} = \cap_{j=1}^{n-i} D_{j,X_i}^0$ (resp. $W_i^{pr} = \cap_{j=i+1}^{n} D_{j,Y_i}^1$). Therefore the inverse image of X_i^0 (resp. Y_i^0) under the morphism $X_{i+1} \to X_i$ (resp. $Y_{i+1} \to Y_i$) is isomorphic to X_i^0 (resp. Y_i^0). We identify X_i^0 (resp. Y_i^0) as an open set of X_{i+1} (resp. Y_{i+1}) via this isomorphism. Let $E_i^0 = X_{i+1}^0 - X_i^0$ (resp. $F_i^0 = Y_{i+1}^0 - Y_i^0$).

In general, for a system of coordinate $(y_1, \dots, y_k)$ of $(\mathbf{A}^1)^k$, we consider the same procedure as before and the resultant variety X and Y are denoted by $X(y_1, \dots, y_k)$ and $Y(y_1, \dots, y_k)$, respectively. The divisors corresponding to D_Y, $D_{i,X}^\epsilon$ and $D_{i,Y}^\epsilon$ are denoted by $D_Y(y_1, \dots, y_k)$, $D_{y_i,X}^\epsilon(y_1, \dots, y_k)$ and $D_{y_i,Y}^\epsilon(y_1, \dots, y_k)$, respectively. The following propositions can be easily checked.

Proposition 2.1.
1. *The divisors D_{j,X_i}^0 ($j = n-i+2, \dots, n$) and D_{j,X_i}^1 ($j = n-i+1, \dots, n$) transversally intersect with Z_i^{pr}. The divisors D_{j,Y_i}^1 ($j = 1, \dots, i-1$) and D_{j,Y_i}^0 ($j = 1, \dots, i$) transversally intersect with W_i^{pr}.*
2. *The divisor D_{j,X_i}^1 does not intersect with Z_i^{pr} for $j = 1, \dots, n-i$. The divisor D_{n-i+1,X_i}^0 does not intersect with Z_i^{pr}.*

3. *The divisor* D^0_{j,Y_i} *does not intersect with* W^{pr}_i *for* $j = i+1, \dots, n$. *The divisor* D^1_{i,Y_i} *does not intersect with* W^{pr}_i.

Proposition 2.2. *The variety* Z^{pr}_i *(resp.* W^{pr}_i*) is naturally identified with* $X(x_{n-i+1}, \dots, x_n)$ *(resp.* $Y(x_1, \dots, x_i)$*).*

1. *Under the identification* $Z^{pr}_i = X(x_{n-i+1}, \dots, x_n)$, *the intersection* $D^0_{x_j,X_i} \cap Z^{pr}_i$ *(resp.* $D^1_{x_j,X_i} \cap Z^{pr}_i$*) is identified with* $D^0_{x_j,X}(x_{n-i+1}, \dots, x_n)$ *for* $n-i+2 \le j \le n$ *(resp.* $D^1_{x_j,X}(x_{n-i+1}, \dots, x_n)$ *for* $n-i+1 \le j \le n-1$*).*
2. *Under the identification* $W^{pr}_i = Y(x_1, \dots, x_i)$, *the intersection* $D^0_{x_j,Y_i} \cap W^{pr}_i$ *(resp.* $D^1_{x_j,Y_i} \cap W^{pr}_i$*) is identified with* $D^0_{x_j,Y}(x_1, \dots, x_i)$ *for* $2 \le j \le i$ *(resp.* $D^1_{x_i,Y}(x_1, \dots, x_i)$ *for* $1 \le j \le i-1$*).*

Corollary 2.3.

$$\begin{aligned} Z^0_i =& Z^{pr}_i - \cup_{j=n-i+2}^n D^0_{j,X_i}(x_{n-i+1}, \dots, x_n) \\ & - \cup_{j=n-i+1}^{n-1} D^1_{j,X_i}(x_{n-i+1}, \dots, x_n), \\ W^0_i =& W^{pr}_i - \cup_{j=2}^i D^0_{j,Y_i}(x_1, \dots, x_i) - \cup_{j=1}^{i-1} D^1_{j,Y_i}(x_1, \dots, x_i). \end{aligned}$$

Let E_i (resp. F_i) be the exceptional divisor of the blowing up $X_{i+1} \to X_i$ (resp. $Y_{i+1} \to Y_i$). The the morphism $\pi_i : E_i \to Z^{pr}_i$ ($\tau_i : F_i \to W^{pr}_i$) is a $\mathbf{P}^{n-i-1}$-bundle and the intersections $D^0_{j,X_{i+1}} \cap E_i$ $(1 \le j \le n-i)$(resp. $D^1_{j,Y_{i+1}} \cap F_i$ $(i+1 \le j \le n)$) are horizontal families of independent hyperplanes for the morphism $E_i \to Z^{pr}_i$ (resp. $F_i \to W^{pr}_i$). Therefore $E_i - \cup_{j=2}^{n-i} D^0_{j,X_{i+1}}$ (resp. $F_i - \cup_{j=i+1}^{n-1} D^1_{j,X_{i+1}}$) is a $\mathbf{A}^1 \times (\mathbf{G}_m)^{n-2-i}$-bundle over Z^{pr}_i (resp. W^{pr}_i). By Corollary 2.3, the morphism π_i (resp. τ_i) induces a morphism $E^0_i \to Z^0_i$ (resp. $F^0_i \to W^0_i$), which is also denoted by π_i (resp. τ_i).

We introduce an open set U_i of Z^0_i and its neighborhood N_i. We use them and their blowing ups to compute the de Rham cohomology of X^0 in Proposition 2.7.

By Proposition 2.2 and Corollary 2.3, Z^0_i contains an openset U_i defined by

$$\begin{aligned} U_i = \{(x_{n-i+1}, \dots, x_n) \,| x_k \neq 0 (\text{ for } n-i+1 \le k \le n), \\ x_k \neq 1 (\text{ for } n-i+1 \le k \le n-1)\}. \end{aligned}$$

We give a description of the restriction of π_i to U_i. Let

$$N_i = \{(x_1, \dots, x_n) \mid (x_{n-i+1}, \dots, x_n) \in U_i\}.$$

The variety U_i can be identified with the closed subvariety $\{(x_1, \dots, x_n) \in N_i \mid x_1 = \dots = x_{n-i} = 0\}$ of N_i. Since N_i does not intersect with $Z_0, \dots, Z_{i-1}$ in X_0, N_i can be identified with a subvariety in X_i via the blowing up procedure. Moreover we have $N_i \cap Z^0_i = U_i$ in X_i. Let $Bl_{U_i}(N_i)$ be the blowing up of N_i along the center U_i. Then $Bl_{U_i}(N_i)$ can be identified with an open set of X_{i+1}, and we have the following cartesian diagram.

$$\begin{array}{ccc} X_{i+1} & \rightarrow & X_i \\ \cup & & \cup \\ Bl_{U_i}(N_i) & \rightarrow & N_i \end{array}$$

Let $Bl_{U_i}(N_i)^0 = Bl_{U_i}(N_i) - \cup_{j=2}^{n-i} D^0_{j,X_{i+1}} - \cup_{j=1}^{n-i} D^1_{j,X_{i+1}}$. Then $Bl_{U_i}(N_i)^0 = X^0_{i+1} \cap Bl_{U_i}(N_i)$. We put $E^0_{i,U_i} = Bl_{U_i}(N_i)^0 \cap E^0_i$. Then we have the following commutative diagram.

$$\begin{array}{ccc} E^0_i & \rightarrow & Z^0_i \\ \uparrow & & \uparrow \\ E^0_{i,U_i} & \rightarrow & U_i \end{array}$$

Proposition 2.4. *We introduce coordinates ξ_i with $\xi_i x_2 = x_i$ for $i = 1, 3, 4, \ldots, n-i$.*

1. *The variety $Bl_{U_i}(N_i)^0$ is isomorphic to*

$$\begin{aligned} \{(\xi_1, &\xi_3, \ldots, \xi_{n-i}, x_2, x_{n-i+1}, \ldots, x_n) \mid \xi_i \neq 0 (\text{ for } 3 \leq i \leq n-i), \\ &x_2\xi_i \neq 1 (\text{ for } 3 \leq i \leq n-i \text{ or } i = 1), x_i \neq 0 (\text{ for } n-i+1 \leq i \leq n), \\ &x_i \neq 1 (\text{ for } n-i+1 \leq i \leq n-1 \text{ or } i = 2)\}. \end{aligned}$$

2. *The variety E^0_{i,U_i} is isomorphic to the subvariety of $Bl_{U_i}(N_i)^0$ defined by $x_2 = 0$ under the coordinate given as above. More explicitly, it is isomorphic to*

$$\begin{aligned} \{(\xi_1, \xi_3, \ldots, \xi_{n-i}, x_{n-i+1}, \ldots, x_n) \mid \xi_i \neq 0 (\text{ for } 3 \leq i \leq n-i), \\ x_i \neq 0 (\text{ for } n-i+1 \leq i \leq n), x_i \neq 1 (\text{ for } n-i+1 \leq i \leq n-1)\}. \end{aligned}$$

We also use the following proposition for the computation of de Rham cohomology in the next subsection.

Proposition 2.5. *The open set $Bl_{U_i}(N_i)^0 - E^0_{i,U_i}$ is isomorphic to*

$$\begin{aligned} \mathbf{U} = \{(x_1, \ldots, x_n) \mid & x_i \neq 0 (\text{ for } 2 \leq i \leq n), \\ & x_i \neq 1 (\text{ for } 1 \leq i \leq n-1)\}. \end{aligned}$$

2.2. **Computation of de Rham cohomology.** In this section, we compute the de Rham cohomology of X^0 and Y^0. Let K be the field of definition $\mathbf{Q}$.

We define a set $\mathcal{S}$ of the differential forms on X^0_0 by

$$\mathcal{S} = \{\frac{dx_i}{x_i} \ (i = 2, \ldots, n), \frac{dx_i}{x_i - 1} \ (i = 1, \ldots, n-1)\}$$

Let $1 \leq i_1 < \cdots < i_k \leq n$, $\dfrac{dx_{i_r}}{x_{i_r} - e_r}$ be an element in $\mathcal{S}$ for $1 \leq r \leq k$. A differential form

$$\omega = \omega(i_1, \ldots, i_k; e_1, \ldots, e_k) = \frac{dx_{i_1}}{x_{i_1} - e_1} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k} - e_k} \tag{2.1}$$

is said to begin with type 1 (resp. end by type 0) if $e_1 = 1$ (resp. $e_k = 0$). The vector space of k-forms generated by $\omega(i_1, \ldots, i_k; e_1, \ldots, e_k)$ with $e_1 = 1$

(resp. $e_1 = 1$ and $e_k = 0$) is denoted by V_1^k (resp. V_{10}^k). If $k = 0$, we define $V_1^0 = V_{10}^0 = K$. The main theorem of this section is as follows.

Theorem 2.6. 1. $H_{DR}^k(X^0) \simeq V_1^k$.
2. $H_{DR}^k(Y^0) \simeq V_{10}^k$.

Let $V_{k,p}$ be the subspace of k-forms $H_{DR}^k(\mathbf{U})$ of $\mathbf{U}$ generated by $\omega(i_1, \dots, i_k; e_1, \dots, e_k)$, where (1) there exists r such that $i_r = p$ and $e_r = 1$, and (2) $e_q = 0$ for $q < r$. For a formal sum $a = \sum_{J \subset [1,n]} a_J J$ ($a_J \in \mathbf{Q}$) of simplices J in $[1, n]$, we define $\omega^0(a, x)$ by

$$\omega^0(a, x) = \sum_J a_J \prod_{j \in J} \frac{dx_j}{x_j}. \tag{2.2}$$

The boundary operator is written as ∂ and the k-th face operator is denoted by "$-\{k\}$". For $2 \le q < p \le n+1$, we define a subspace $V_{k,p,q}$ by the subspace of k-form generated by $\omega^0(\partial J, x) \wedge \frac{dx_p}{x_p - 1} \wedge \eta$ (resp. $\omega^0(\partial J, x)$), if $p \le n - 1$ (resp. $p = n$ or $p = n + 1$) where $J \subset [2, q]$ and η a product of $\dfrac{dx_j}{x_j - e_j} \in \mathcal{S}$ with $j > p$. Since $V_{k,p,2} = V_{k,p}$ and $V_1^k = V_{k,1} \oplus V_{k,2} \oplus \oplus_{p=3}^{n-1} V_{k,p,2}$, the following proposition implies Theorem 2.6.1.

Proposition 2.7. 1. *The natural map $H^k(X_i^0) \to H^k(\mathbf{U})$ is injective.*
2. *Under the above injection, $H^k(X_i^0)$ ($i = 1, \dots, n - 1$) is identified with*

$$\oplus_{p=1}^{n-i+1} V_{k,p} \oplus \oplus_{p=n-i+2}^{n+1} V_{k,p,n-i+1}.$$

Proof. We prove the proposition by induction of $\dim X$ and i. For $i = 1$, we can prove the proposition directly. We consider the following commutative diagram whose rows are exact:

$$\begin{array}{ccccc}
H^k(X_{i+1}^0) & \to & H^k(X_i^0) & \xrightarrow{r} & H^{k-1}(E_i^0) \\
\downarrow & & \alpha \downarrow & & \downarrow \beta \\
H^k(Bl_{U_i}(N_i)^0) & \to & H^k(\mathbf{U}) & \xrightarrow{r} & H^{k-1}(E_{i,U_i}^0)
\end{array}$$

By the hypothesis of the induction, α is injective. By using the Leray spectral sequence and inductive hypothesis, β is also injective. By Proposition 2.5 and Proposition 2.4 $H^k(\mathbf{U})$, $H^k(E_{i,U_i}^0)$ and $H^k(E_i^0)$ are generated by $\mathcal{B}_{\mathbf{U}}$, $\mathcal{B}_{E_{i,U_i}^0}$ and $\mathcal{B}_{E_i^0}$, where

$$\mathcal{B}_{\mathbf{U}} = \{\omega(i_1, \dots, i_k; e_1, \dots, e_k) \mid \frac{dx_{i_k}}{x_{i_k} - e_i} \in \mathcal{S}\},$$

$$\mathcal{S}^{(i)} = \{\frac{dx_i}{x_i} \ (i = n - i + 1, \dots, n), \frac{dx_i}{x_i - 1} \ (i = n - i + 1, \dots, n - 1)\},$$

$$\mathcal{B}_{E_{i,U_i}^0} = \{\omega^0(J, \xi) \wedge \omega(j_1, \dots, j_l; e_1, \dots, e_l) \mid J \subset [3, n - i], \frac{dx_{j_k}}{x_{j_k} - e_i} \in \mathcal{S}^{(i)}\},$$

$$\mathcal{B}_{E_i^0} = \{\omega^0(J, \xi) \wedge \omega(j_1, \dots, j_l; e_1, \dots, e_l) \in \mathcal{B}_{E_{i,U_i}^0} \mid e_1 = 1\}.$$

Here we used notations (2.1) and (2.2). The horizontal homomorphisms r are obtained by Poincare residue with respect to the divisor $x_2 = 0$. We prove that $r : H^k(X_i^0) \to H^k(E_i^0)$ is surjective and the kernel of r is equal to $\oplus_{p=1}^{n-i} V_{k,p} \oplus \oplus_{p=n-i+1}^{n+1} V_{k,p,n-i}$. Let $T = (i_1, \dots, i_l; e_1, \dots, e_l)$ with $i_1, \dots, i_l \in [p+1, n]$. We use the notation $\omega(T)$ in (2.1). We define $V_{k,p}^T$ and $V_{k,p,q}^T$ as the subspace of $V_{k,p}$ and $V_{k,p,q}$ generated by $\omega^0(J,x) \wedge \dfrac{dx_p}{x_p - 1} \wedge \omega(T)$ and $\omega^0(\partial J, x) \wedge \dfrac{dx_p}{x_p - 1} \wedge \omega(T)$ with $J \in [2,q]$, respectively. Then we have $V_{k,p} = \oplus_T V_{k,p}^T$ and $V_{k,p,q} = \oplus_T V_{k,p,q}^T$ for $p > n-i$. We define $V_{k,p,n-i}^{T,\xi}$ by the subspace of $H^k(E_{i,U_i}^0)$ generated by

$$\{\omega^0(J,\xi) \wedge \frac{dx_p}{x_p - 1} \wedge \omega(T) \mid J \subset [3, n-i]\}.$$

We compute the residue r with respect to $x_2 = 0$ by the relation $\dfrac{dx_k}{x_k} = \dfrac{d\xi_k}{\xi_k} + \dfrac{dx_2}{x_2}$ for $k = 3, \dots, n-i$. It is easy to see that $r(V_{k,p,n-i+1}^T) \subset V_{k,p,n-i}^{T,\xi}$ for $p > n-i+1$. For $J \subset [2, n-i]$, we have

(2.3)

$$r(\omega^0(J,x) \wedge \frac{dx_p}{x_p - 1} \wedge \omega(T)) = \begin{cases} \omega^0(\partial J, \xi) \wedge \frac{dx_p}{x_p - 1} \wedge \omega(T) & \text{if } 2 \notin J \\ \omega^0(J - \{2\}, \xi) \wedge \frac{dx_p}{x_p - 1} \wedge \omega(T) & \text{if } 2 \in J \end{cases}$$

and $r(V_1) = \cdots = r(V_{n-i}) = 0$. Therefore it is enough to prove that $r : V_{k,n-i+1}^T \to V_{k,n-i+1,n-i}^{T,\xi}$ and $r : V_{k,p,n-i+1}^T \to V_{k,p,n-i}^{T,\xi}$ $(p > n-i+1)$ are surjective and

$$ker(r : V_{k,n-i+1}^T \to V_{k,n-i+1,n-i}^{T,\xi}) = V_{k,n-i+1,n-i}^T,$$
$$ker(r : V_{k,p,n-i+1}^T \to V_{k,p,n-i}^{T,\xi}) = V_{k,p,n-i}^T.$$

(1) **The morphism** $r : V_{k,n-i+1}^T \to V_{k,n-i+1,n-i}^{T,\xi}$. For $J \subset [3, n-i]$, $(\omega^0(J \cup \{2\}, x) \wedge \dfrac{dx_{n-i+1}}{x_{n-i+1} - 1} \wedge \omega(T)) = \omega^0(J,\xi) \wedge \dfrac{dx_{n-i+1}}{x_{n-i+1} - 1} \wedge \omega(T)$. Therefore r is surjective. Suppose that the element

$$\eta = \sum_{J \subset [2, n-i]} a_J \omega^0(J,x) \wedge \frac{dx_{n-i+1}}{x_{n-i+1} - 1} \wedge \omega(T) \tag{2.4}$$

is contained in the kernel of r. For a subset $K \subset [2, n-i]$ such that $2 \in K$, the element $\tau = \omega^0(\partial K, x) \wedge \dfrac{dx_{n-i+1}}{x_{n-i+1} - 1} \wedge \omega(T)$ in $V_{k,n-i+1,n-i}^T$ contains only one term $\omega^0(J,x) \wedge \dfrac{dx_{n-i+1}}{x_{n-i+1} - 1} \wedge \omega(T)$ such that $2 \notin J$, and in this case, $J = K - \{2\}$. Therefore there exist an element $\delta \in V_{k,n-i+1,n-i}^T$ such that $\eta' = \eta - \delta$ contains no term with $2 \notin J$ in expression (2.4). The relation $r(\eta') = 0$ and formula (2.3) implies $\eta' = 0$.

(2) **The morphism** $r : V^T_{k,p,n-i+1} \to V^{T,\xi}_{k,p,n-i}$. Let $2 \in K \subset [2, n-i+1]$. Then we have

$$r(\omega^0(\partial K, x) \wedge \frac{dx_p}{x_p - 1} \wedge \omega(T)) \tag{2.5}$$
$$= \begin{cases} \omega^0(K - \{2, n-i+1\}, \xi) \wedge \frac{dx_p}{x_p-1} \wedge \omega(T) & (\text{ if } n-i+1 \in K) \\ 0 & (\text{ if } n-i+1 \notin K), \end{cases}$$

and the surjectivity follows from this equality. Suppose that the element

$$\eta = \sum_{J \subset [2,n-i+1]} a_J \omega^0(\partial J, x) \wedge \frac{dx_p}{x_p - 1} \wedge \omega(T) \tag{2.6}$$

is contained in the kernel of r. Let $2 \in K \subset [2, n-i+1]$. The boundary $\partial K = \sum_J b_J J$ contains only one term with $2 \notin J$. By using the relation $\sum_J b_J \omega^0(\partial J, x) \wedge \frac{dx_p}{x_p-1} \wedge \omega(T) = 0$, we may assume that there is no term with $2 \notin J$ in the expression of (2.6). Then by formula (2.5) and $r(\eta) = 0$, there is no term with $n-i+1 \in J$ in the expression of (2.6). Therefore $\eta \in V^T_{k,p,n-i}$. $\square$

We completed the proof of Theorem 2.6.1. The proof of Theorem 2.6.2 is similar. We omit the proof of Theorem 2.6.2.

2.3. **Proper transform of S-diagonal varieties.** We define divisors B_i in $(\mathbf{A}^1)^n$ by $B_0 = \{x_1 = 0\}$, $B_n = \{x_n = 1\}$ and $B_i = \{x_i = x_{i+1}\}$ for $i = 1, \dots, n-1$. Let S be a subset of $[0, n]$ such that $S \neq [0, n]$. We define $B_S = \cap_{i \in S} B_i$ and $B_\emptyset = X_0$. Then $\dim B_S = \#\bar{S} - 1$, where $\bar{S} = [0, n] - S$. Set $\bar{S} = \{i_1, \dots, i_s\}$. Then a coordinate of B_S is given by $x_{i_2}, \dots x_{i_s}$, since

$$x_{i_1+1} = \cdots = x_{i_2}, \dots, x_{i_{s-1}+1} = \cdots = x_{i_s}.$$

The proper transform of B_i in Y is denoted by B_i^{pr}. Then it is easy to see that

$$B_0^{pr} \simeq Y(x_2, \dots, x_n). \tag{2.7}$$

Let us define divisors $\tilde{B}_i$ in Y by

$$\tilde{B}_0 = B_0^{pr} \cup E_0^{pr} \cup \cdots \cup E_{n-2}^{pr},$$
$$\tilde{B}_i = B_i^{pr},$$
$$\tilde{B}_n = B_n^{pr} \cup F_0^{pr} \cup \cdots \cup F_{n-2}^{pr},$$

where E_i^{pr} (resp. F_i^{pr}) is the proper transform of E_i (resp. F_i) in Y. We put $\tilde{B}_S = \cap_{i \in S} \tilde{B}_i$. We define $B_i^{pr,0} = B_i^{pr} \cap Y^0$, $\tilde{B}_i^0 = \tilde{B}_i \cap Y^0$ and $\tilde{B}_S^0 = \tilde{B}_S \cap Y^0$. Then via isomorphism (2.7), we have $B_0^{pr,0} \simeq Y^0(x_2, \dots, x_n)$. We can easily check the following lemma.

Lemma 2.8. 1. *There exists natural homomorphisms*

$$\begin{cases} E_i^{pr} \to Y(x_{n-i+1}, \dots, x_n) \\ F_j^{pr} \to Y(x_1, \dots, x_j), \end{cases} \tag{2.8}$$

which are birational to the morphisms $E_i \to Z_i^{pr}$ and $F_j \to W_j^{pr}$ in §2.1.

2. *Let $E_i^{pr,0} = E_i^{pr} \cap Y^0$ (resp. $F_j^{pr,0} = F_j^{pr} \cap Y^0$). Then the morphisms (2.8) induce morphisms*

$$E_i^{pr,0} \to Y^0(x_{n-i+1}, \dots, x_n)$$
$$F_j^{pr,0} \to Y^0(x_1, \dots, x_j)$$

and they are trivial $\mathbf{A}^{n-i-1}$-bundle and trivial $\mathbf{A}^{n-j-1}$-bundle, respcetivly. Moreover $\eta_1 = x_1/x_2, \dots, \eta_{n-i-1} = x_{n-i-1}/x_{n-i}$ gives a trivialization of $\mathbf{A}^{n-i-1}$-bundle $E_i^{pr,0} \to Y^0(x_{n-i+1}, \dots, x_n)$.

3. *The intersections $E_p^{pr} \cap E_i^{pr,0}$ $(p > i)$ and $B_q^{pr} \cap E_i^{pr,0}$ $(0 \leq q < n - i - 1)$ are horizontal divisors with respect to the morphism $E_i^{pr,0} \to Y^0(x_{n-i+1}, \dots, x_n)$. The defining equations of them are*

$$\begin{cases} E_i^{pr,0} \cap E_{n-2}^{pr} : \eta_2 = 0, \dots, E_i^{pr,0} \cap E_{i+1}^{pr} : \eta_{n-i-1} = 0, \\ E_i^{pr,0} \cap B_0^{pr} : \eta_1 = 0, E_i^{pr,0} \cap B_1^{pr} : \eta_1 = 1, \\ E_i^{pr,0} \cap B_2^{pr} : \eta_2 = 1, \dots, E_i^{pr,0} \cap B_{n-i-1}^{pr} : \eta_{n-i-1} = 1. \end{cases} \tag{2.9}$$

4. *Using the coordinate in (2.9) of Lemma 2.8.3, the projection*

$$(\eta_1, \eta_2, \dots, \eta_{n-i-1}) \mapsto (\eta_2, \dots, \eta_{n-i-1})$$

gives a morphism $\pi_i : E_i^{pr,0} \to E_i^{pr,0} \cap B_0^{pr,0}$. Moreover the morphisms π_i patched together into the map $\pi : \tilde{B}_0^0 \to B_0^{pr,0}$. The morphism π is homotopy equivalent.

Definition 2.9. *Via the isomorphism 2.7, the divisor of $Y^0(x_2, \dots, x_n)$ corresponding to the divisor $\tilde{B}_i^0$ is denoted by $\tilde{B}_i^0(x_2, \dots, x_n)$. In general, for a set of coordinates $(y_1, \dots, y_m)$, the divisor of $Y^0(y_1, \dots, y_m)$ corresponding to $\tilde{B}_i^0$ is denoted by $\tilde{B}_i^0(y_1, \dots, y_m)$. Then we have*

$$\tilde{B}_1^0(x_2, \dots, x_n) = \cup_{i=0}^{n-2} E_i^{pr} \cap B_0^{pr,0},$$
$$\tilde{B}_i^0(x_2, \dots, x_n) = B_i^{pr} \cap B_0^{pr,0},$$
$$\tilde{B}_n^0(x_2, \dots, x_n) = \tilde{B}_n \cap B_0^{pr,0}.$$

For a set $S' \subset [1, n]$, we define $\tilde{B}_{S'}^0(x_2, \dots, x_n) = \cap_{i \in S'} \tilde{B}_i^0(x_2, \dots, x_n)$.

The following proposition is a direct consequence of the definition of $\pi : \tilde{B}_0^0 \to B_0^{pr,0}$ defined in Lemma 2.8.4.

Proposition 2.10. *Let $S \subset [0, n]$ and $0 \in S$. We put $S' = S - \{0\}$.*

1. $\pi(\tilde{B}_S^0) = \tilde{B}_{S'}^0(x_2, \dots, x_n)$.
2. *We have the following commutative diagram, where the horizontal arrows are homotopy eqauivalent.*

$$\begin{tikzcd} \tilde{B}_0^0 \arrow[r, "\pi"] & B_0^{pr,0} \\ \tilde{B}_S^0 \arrow[u] \arrow[r, "\text{homotopy equivalent}"'] & \tilde{B}_{S'}^0(x_2, \dots, x_n) \arrow[u] \end{tikzcd}$$

By using Proposition 2.10 successively, we have the following corollary.

Corollary 2.11. *Let $S \subset [0,n]$ such that $0 \in S$ and $\bar{S} = \{i_1, \dots, i_s\}$. The variety $\tilde{B}_S^0$ is homotopy equivalent to*

$$E_{n-i_1}^0 \cap B_1^{pr,0} \cap \cdots \cap B_{i_1-1}^{pr,0} \cap (\cap_{i_1<j<i_s, j\in S} B_j^{pr,0}) \cap B_{i_s+1}^{pr,0} \cap \cdots \cap B_n^{pr,0} \cap F_{i_s-1}^0 \tag{2.10}$$

Via the isomorphism

$$E_{n-i_1}^0 \cap B_1^{pr,0} \cap \cdots \cap B_{i_1-1}^{pr,0} \cap B_{i_s+1}^{pr,0} \cap \cdots \cap B_n^{pr,0} \cap F_{i_s-1}^0 \simeq Y^0(x_{i_1+1}, \dots, x_{i_s}),$$

(2.10) is isomorphic to

$$\tilde{B}_{\hat{S}}^0(x_{i_1+1}, \dots, x_{i_s}) = \cap_{j\in\hat{S}} \tilde{B}_j^0(x_{i_1+1}, \dots, x_{i_s}) \subset Y^0(x_{i_1+1}, \dots, x_{i_s}),$$

where $\hat{S} = [i_1, i_s] \cap S$. $\tilde{B}_{\hat{S}}^0(x_{i_1+1}, \dots, x_{i_s})$ is identified with $Y^0(x_{i_2}, x_{i_3}, \dots, x_{i_s})$ and is denoted by $B_S^{pr,0}$.

All the varieties in this paper are defined over $K = \mathbf{Q}$. For a smooth variety S over K, the de Rham complex $\Omega_{S/K}^\bullet$ is denoted by $K_{S,DR}$. Then by the definition of de Rham cohomology, we have $H_{DR}^i(S/K) = H^i(K_{S,DR})$. More generally, for a normal corssing variety $V = \cup_i V_i$, we define the de Rham complex $K_{V,DR}$ of V by the complex

$$\oplus_{\#I=1} K_{V_I,DR} \to \oplus_{\#I=2} K_{V_I,DR} \to \cdots,$$

where $V_I = \cap_{i\in I} V_i$.

Corollary 2.12. *The morphism $\tilde{B}_S^0 \to B_S^{pr,0}$ in Corollary 2.11 gives the following quasi-isomorphism*

$$K_{B_S^{pr,0},DR} \to K_{\tilde{B}_S^0,DR}. \tag{2.11}$$

Let

$$B_S^0 = \{(x_{i_2}, \dots, x_{i_s}) \mid x_{i_2} \neq 1, x_{i_3} \neq 0, 1, \dots, x_{i_{s-1}} \neq 0, 1, x_{i_s} \neq 0\}$$

Then we have $B_S^0 \subset B_S^{pr,0}$ and the induced map $H^k(B_S^{pr,0}) \subset H^k(B_S^0)$ is injective. By Theorem 2.6, we have the following corollary.

Corollary 2.13. *Under the notation of (2.1), the cohomology group $H_{DR}^k(B_S^{pr,0})$ is generated freely by*

$$\omega(j_1, \dots, j_p; \epsilon_1, \dots, \epsilon_k),$$

where $\{j_1 < \cdots < j_k\} \subset \{i_2, \dots, i_s\}$, $\epsilon_p \in \{0,1\}$ for $p = 2, \dots, k-1$, $\epsilon_1 = 1$ and $\epsilon_k = 0$.

Proposition 2.14. *Let $S \subset [0,n]$ and put $\bar{S} = \{i_1, \dots, i_s\}$. We define $S^{(m)} \subset [0,n]$ such that $\bar{S}^{(m)} = \{i_1, \dots, i_{m-1}, i_{m+1}, \dots, i_s\}$. We put*

$$\begin{aligned}
B_{S^{(0)}}^{00} =& \{(x_{i_3}, \dots, x_{i_s}) \mid x_{i_3} \neq 0, 1, \dots, x_{i_{s-1}} \neq 0, 1, x_{i_s} \neq 0\}, \\
B_{S^{(m)}}^{00} =& \{(x_{i_2}, \dots, x_{i_{m-1}}, x_{i_{m+1}}, \dots, x_{i_s}) \mid \\
& x_{i_2} \neq 1, x_{i_2} \neq 0, 1, \dots, x_{i_{s-1}} \neq 0, 1, x_{i_s} \neq 0\}, \\
B_{S^{(s)}}^{00} =& \{(x_{i_2}, \dots, x_{i_{s-1}}) \mid x_{i_2} \neq 1, x_{i_3} \neq 0, 1, \dots, x_{i_{s-1}} \neq 0, 1\},
\end{aligned}$$

for $m = 1, \ldots, s-1$. Then we have the following commutative diagram for $m = 0, \ldots, s$.

$$\begin{array}{ccc} B^{00}_{S^{(m)}} & \hookrightarrow & B^{0}_{S} \\ \downarrow & & \downarrow \\ B^{pr,0}_{S^{(m)}} & & B^{pr,0}_{S} \\ \uparrow & & \uparrow \\ \tilde{B}^{0}_{S^{(m)}} & \hookrightarrow & \tilde{B}^{0}_{S} \end{array}$$

2.4. **Relative de Rham cohomology groups.** We use the same notations $\tilde{B}^0_i$, $\tilde{B}^0_S$, $B^{pr,0}_i$ and $B^{pr,0}_S$ as in the last subsection. Then the divisor $\mathbf{B}^0 = \cup_{i=0}^n \tilde{B}^0_i$ of Y^0 is normal crossing. The complement of $\mathbf{B}^0$ in Y^0 is denoted by V and the natural inclusions $V \to Y^0$ and $\mathbf{B}^0 \to Y^0$ are denoted by j and i, respectively.

$$V \xrightarrow{j} Y^0 \xleftarrow{i} \mathbf{B}^0$$

For subsets S, T of $[0, n]$, such that $S \subset T \neq [0, n]$, the natural closed immersion $\tilde{B}^0_T \to \tilde{B}^0_S$ is defined. Let $j_! K_{V,DR}$ be the cone $Cone(K_{Y^0,DR} \to K_{\mathbf{B}^0,DR})$.

Using the natural restriction morphism we have a complex

$$\begin{aligned}(2.12)\qquad \oplus_{\substack{\#S=* \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR} = (\oplus_{\substack{\#S=0 \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR} \to \oplus_{\substack{\#S=1 \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR} \\ \cdots \to \oplus_{\substack{\#S=n \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR})\end{aligned}$$

Proposition 2.15. *The natural morphism*

$$j_! K_{V,DR} \to \oplus_{\substack{\#S=* \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR}$$

is a quasi-isomorphism.

We have the spectral sequence

$$(2.13)\qquad E_1^{p,q} = \oplus_{\substack{\#S=p \\ S\subset[0,n]}} H^q(K_{\tilde{B}^0_S,DR}) \Rightarrow H^{p+q}(j_! K_{V,DR})$$

associated to the stupid filtration σ^*:

$$\begin{aligned}\sigma^i(\oplus_{\substack{\#S=* \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR}) = (0 \to \cdots \to 0 \to \oplus_{\substack{\#S=i \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR} \\ \cdots \to \oplus_{\substack{\#S=n \\ S\subset[0,n]}} K_{\tilde{B}^0_S,DR} \to 0)\end{aligned}$$

By quasi-isomorphism (2.11), E_1-term of spectral sequence (2.13) is isomorphic to $E_1^{p,q} = \oplus_{\substack{\#S=p \\ S\subset[0,n]}} H^q_{DR}(B^{pr,0}_S)$. To compute the differential d_1 of the spectral sequence, we use the following commutative diagram, which is a consequence of Proposition 2.14.

$$\begin{array}{ccc} H^q_{DR}(B^{pr,0}_S) & \to & H^q_{DR}(B^{pr,0}_{S^{(m)}}) \\ \downarrow & & \downarrow \\ H^q_{DR}(B^0_S) & \to & H^q_{DR}(B^{0,0}_{S^{(m)}}), \end{array}$$

where $S, S^{(m)}$ are as in Proposition 2.14.

We define the word type of $\omega(j_1, \dots, j_p; \epsilon_1, \dots, \epsilon_k)$ in (2.1) as $(\epsilon_1, \dots, \epsilon_k)$. For a word $W = (\epsilon_1, \dots, \epsilon_k)$, k is called the length of W, which we denote by $\mathrm{len}(W)$. Let $\mathbf{W}_q$ be the set of words $W = (\epsilon_1, \dots, \epsilon_q)$ of length q such that $\epsilon_i = 0, 1$, $\epsilon_1 = 1$ and $\epsilon_q = 0$. Note that $\mathbf{W}_0$ consists of the empty word $()$.

Theorem 2.16. *For spectral sequence (2.13), we have $E_2^{p,k} = 0$ if $p + k \neq n$ and*

$$E_2^{n-k,k} \simeq \oplus_{W \in \mathbf{W}_k} K.$$

As a consequence, we have $H^i(j_! K_{V,DR}) = 0$ for $i \neq n$ and the filtration on $H^n(j_! K_{V,DR})$ induced by spectral sequence (2.13) coincides with the weight filtration.

Proof. The subspace of $H^k_{DR}(B_S^{pr,0})$ generated by differential forms of word type $W = (\epsilon_1, \dots, \epsilon_k)$ is denoted by $H_{S,W}$. Then, by Corollary 2.13, we have

$$H^k_{DR}(B_S^{pr,0}) \simeq \oplus_{W \in \mathbf{W}_k} H_{S,W}.$$

We can choose the local coordinate of B_S^0 as $(x_{i_2}, \dots, x_{i_s})$ if $\bar{S} = \{i_1 < \cdots < i_s\}$. Then the differential d_1 of the spectral sequence $E_1^{p,k}$ of (2.13) preserves the word type and as a consequence, the complex $E_1^{*,k}$ is a direct sum $\oplus_{W \in \mathbf{W}_k} H_W^*$ of the complices H_W^*, where

$$H_W^* : H_W^0 \simeq \oplus_{\substack{\#\bar{S}=n+1 \\ S \subset [0,n]}} H_{S,W} \to H_W^1 \simeq \oplus_{\substack{\#\bar{S}=n \\ S \subset [0,n]}} H_{S,W} \to \cdots$$

Here we put $\bar{S} = [0,n] - S$. Therefore it is enough to prove the following proposition. □

Proposition 2.17. *Let $W \in \mathbf{W}_k$. We have $H^i(H_W^*) = 0$ for $i \neq n-k$ and $H^{n-k}(H_W^*) \simeq K$.*

Proof. Let $\bar{T}, \bar{S}$ be subsets of $[0,n]$ such that $\#\bar{T} = s-1, \#\bar{S} = s$ and $\bar{T} \subset \bar{S}$. Set $\bar{S} = \{i_1, \dots, i_s\}$ and $\bar{T} = \bar{S} - \{i_l\}$. We put $S = [0,n] - \bar{S}$ and $T = [0,n] - \bar{T}$. Then the restriction $i_{TS} : B_T^{pr,0} \to B_S^{pr,0}$ is given as follows.

1. If $l \neq 1, s$, $i_{TS}^*(x_{i_p}) = x_{i_p}$ if $p \neq l$ and $i_{TS}^*(x_{i_l}) = x_{i_{l+1}}$.
2. If $l = 1$, $i_{TS}^*(x_{i_p}) = x_{i_p}$ if $p \neq 2$ and $i_{TS}^*(x_{i_2}) = 0$.
3. If $l = s$, $i_{TS}^*(x_{i_p}) = x_{i_p}$ if $p \neq s$ and $i_{TS}^*(x_{i_s}) = 1$.

For an integer $0 \le i \le n+1$ and $W \in \mathbf{W}_k$, we define the filtration $F^i(H_W^p)$ of H_W^p to be the subspace generated by $H_{S,W}$ for $\bar{S} \subset [i,n]$ and $\#\bar{S} = n+1-p$. Then we can check that the restriction i_{TS} preserves the filtration and F^* is a filtration of complex H_W^*. The differential on $Gr_F^i(H_W^*)$ induced by d_1 is also denoted by d_1. It is enough to prove the following claim for the proof of the proposition.

Claim Let $W \in \mathbf{W}_k$. (1)If $H^j(Gr_F^i(H_W^*)) \neq 0$ then $i = j = n-k$, and (2)$H^{n-k}(Gr_F^{n-k}(H_W^*)) = K$.

We put $s = n+1-p$. We identify $Gr_F^i(H_W^p)$ with the space $\oplus_{\bar{S} \in \mathcal{S}_{i,s}} H_{S,W}$, where

$$\mathcal{S}_{i,s} = \{\bar{S} = \{i < i_2 < \cdots < i_s\} \mid \#\bar{S} = s\}.$$

The set $\mathcal{S}_{i,s}$ is identified with $\{\bar{S}' \mid \bar{S}' \subset [i+1,n], \#\bar{S}' = s-1\}$. The class of $\omega(j_1,\dots,j_k,W) \in H_{S,W}$ in $Gr^i_F(H^p_W)$ is denoted by $(\bar{S}',\tau)$, where $\bar{S}' = \{i_2,\dots,i_s\}$ and $\tau = (j_1,\dots,j_k)$. The set

$$\{(\bar{S}',\tau) \mid \bar{S}' \subset [i+1,n], \tau \subset \bar{S}', \#\bar{S}' = s-1 = n-p, \#\tau = k\}$$

forms a base of $Gr^i_F(H^p_W)$. Let $\bar{T}'$ be an element of $\mathcal{S}_{i,s-1}$ such that $\bar{T}' \subset \bar{S}'$. The element $\bar{S}' - \bar{T}'$ is denoted by i_l. Then the $\bar{T}'$ component of $d_1(\bar{S}',\tau)$ is

1. $(\bar{T}',\tau)$ if $i_l \notin \tau$,
2. $(\bar{T}',\tau')$, where $\tau' = (j_1,\dots,j_{m-1},i_{l+1},j_{m+1},\dots,j_k)$ if $i_l = j_m$, $l \neq s$ and $i_{l+1} \notin \tau$,
3. zero if $l \neq s$, $i_l \in \tau$, and $i_{l+1} \in \tau$, and
4. zero if $l = s$.

We introduce a partial order on the set $\mathcal{T} = \{\tau \subset [i+1,n] \mid \#\tau = k\}$ so that $(j_{11},\dots,j_{1k}) \leq (j_{21},\dots,j_{2k})$ if and only if $j_{1m} \leq j_{2m}$ for $m = 1,\dots,k$. We choose a numbering $\mathcal{T} = \{\tau_1,\dots,\tau_\kappa\}$ on $\mathcal{T}$ such that $\tau_i \leq \tau_j$ implies $i \leq j$. We define $G^t(Gr^i_F(H^p_W))$ to be the subspace of $Gr^i_F(H^p_W)$ generated by $(\bar{S}',\tau_j)$ for $t \leq j$. Then G^* defines a decreasing filtration on $Gr^i_F(H^p_W)$ preserved by the differential d_1. It is easy to see that the complex $Gr^t_G Gr^i_F(H^*_W)$ is exact if $n-i > k$. If $n-i = k$, then $Gr^i_F(H^p_W) = 0$ if $p \neq n-i$ and $Gr^i_F(H^{n-i}_W)$ is a one dimensional vector space generated by $(\bar{S}',\tau) = ([i+1,n],[i+1,n])$. Therefore we have the claim. □

3. Generators of mixed Tate motives

3.1. **Review of results on mixed Tate motives.** In this subsection, we recall the theory of mixed Tate motives and their properties. M.Levine constructed the derived category $DTM_{\mathbf{Q}}$ of mixed Tate motives in [7]. This is an additive category with the following properties.

(1) **Triangulated category.** (See [7] p.19 Definition 2.1.6., p.45 Corollary 3.4.3.) $DTM_{\mathbf{Q}}$ is a triangulated category. The set of distinguished triangles and the shift operator $A \mapsto A[1]$ are specified. A shift operator is an equivalence of category. A triangle is a pair of morphisms and objects $A \to B \to C \to A[1]$, which is denoted by $A \to B \to C \xrightarrow{+1}$ for short. The following axioms are imposed.

1. For a morphism $f : A \to B$, there exists a unique object and morphisms g and δ such that $C \xrightarrow{g} A \xrightarrow{f} B \xrightarrow[+1]{\delta}$ is a distinguished triangle.
2. If $A \to B \to C \xrightarrow{+1}$ is a distinguished triangle, then $B \to C \to A[1] \xrightarrow{+1}$ and $C[-1] \to A \to B \xrightarrow{+1}$ are distinguished triangles.
3. Let $A \to B \to C \xrightarrow{+1}$ be a distinguished triangle and X be an object of $DTM_{\mathbf{Q}}$. Then we have the following long exact sequences:

$$\cdots \to Hom(X,A) \to Hom(X,B) \to Hom(X,C) \to Hom(X,A[1]) \to \cdots$$

$$\cdots \to Hom(C,X) \to Hom(B,X) \to Hom(A,X) \to Hom(C,X[1]) \to \cdots$$

(2) **Geometric objects.**

1. For an object A in $DTM_{\mathbf{Q}}$ and an integer k, the Tate twist $A(k)$ is defined.
2. An object $\mathbf{Q}_X$ of $DTM_{\mathbf{Q}}$ is attached to a variety X over $\mathbf{Q}$ with a stratification $X_0 = X \supset X_1 \supset X_2 \cdots$ satisfying
$$X_i - X_{i-1} = \coprod_{\text{finite}} \mathbf{A}^{m-i},$$
Moreover it is contravariant with respect to X.
3. Let X_1 and X_2 be smooth varieties as above and $X_1 \to X_2$ be a closed embedding of codimension d. Set $U = X_2 - X_1$. Then $\mathbf{Q}_U$ in $DTM_{\mathbf{Q}}$ is defined functorially on U. Moreover there exists a morphism $\mathbf{Q}_{X_1}(d)[2d] \to \mathbf{Q}_{X_2}$ such that $\mathbf{Q}_{X_1}(d)[2d] \to \mathbf{Q}_{X_2} \to \mathbf{Q}_U \overset{+1}{\to}$ is a distinguished triangle.

(3) **Hodge realization.**(See [7] §2.3, p.273.) There exists a realization functor $\mathbf{H} : DTM_{\mathbf{Q}} \to GMHS$ from $DTM_{\mathbf{Q}}$ to the category of graded objects of mixed Hodge structures with the following properties. The degree i-part of $\mathbf{H}$ is denoted by $\mathbf{H}^i$.

1. For a variety U as in (2), $\mathbf{H}^i(\mathbf{Q}_U) = (H^i_B(U, \mathbf{Q}), H^i_{DR}(U, \mathbf{Q}))$.
2. For a distinguished triangle $A \to B \to C \overset{+1}{\to}$, we have a long exact sequence:
$$\cdots \to \mathbf{H}^i(A) \to \mathbf{H}^i(B) \to \mathbf{H}^i(C) \to \mathbf{H}^{i+1}(A) \to \cdots$$

(5)**Relations to the K-groups.** The following equality holds:
$$Hom_{DTM_{\mathbf{Q}}}(\mathbf{Q}, \mathbf{Q}(i)[1]) = K_{2i-1}(\mathbf{Q}) \otimes \mathbf{Q}, \text{ for } i = 1, 2, \ldots$$
$$Hom_{DTM_{\mathbf{Q}}}(\mathbf{Q}, \mathbf{Q}(i)[2]) = 0.$$

(4)**Weight filtration.** We consider objects with weight filtration. Weight filtration of a object A is a sequence of morphisms $W_iA \to A$, $W_iA \to W_{i+1}A$ with the following properties.

1. For a sufficiently big i (resp. small i), $W_iA \to A$ is an isomorphism (resp. zero map).
2. The cone of $W_{i-1}A \to W_iA$ is an object of $DTM_{\mathbf{Q}}$ generated by $\mathbf{Q}(-i)$ in $DTM_{\mathbf{Q}}$.

Remark 3.1. *In this paper, W_i denotes the $2i$-th weight filtration to simplify the notation.*

An object A is called an abelian object if $Cone(W_{i-1} \to W_i)[1]$ is isomorphic to $\mathbf{Q}(-i)^{r_i}$ for all i. According to [8], the full subcategory A_{TM} of abelian object in $DTM_{\mathbf{Q}}$ is an abelian category.

By the definition of W_i and the compatibility with the realization functor, we have $W_i\mathbf{H}^i(A) = \mathbf{H}(W_iA)$. Therefore A is an abelian object if and only if $\mathbf{H}^i(A) = 0$ if $i \neq 0$. The following proposition is a direct consequence of Theorem 2.16.

Proposition 3.2. *Let $j_!\mathbf{Q}_V$ be the cone $Cone(\mathbf{Q}_{Y^0} \to \mathbf{Q}_{\mathbf{B}^0})$. Then $j_!\mathbf{Q}_V[n]$ is an abelian object.*

3.2. **Comparison of the extension groups.** In this subsection, we compare extension group of mixed motives and that of mixed Hodge structures for certain Tate structures. We define a group homomorphism

$$ch : Ext^1_{MHS}(\mathbf{Z}, \mathbf{Z}(1)) \overset{\simeq}{\to} \mathbf{C}^\times$$

as follows. Let

$$u_{MHS} : 0 \to \mathbf{Z}(1) \to M_{\mathbf{Z}} \overset{\pi}{\to} \mathbf{Z} \to 0 \tag{3.1}$$

be an extension of mixed Hodge structures. Set $M_{\mathbf{C}} = M_{\mathbf{Z}} \otimes \mathbf{C}$. Then the natural map $F^0(M_{\mathbf{C}}) \to \mathbf{Z} \otimes \mathbf{C}$ is an isomorphism. Let ω be the element in $F^0(M_{\mathbf{C}})$ corresponding to 1 via this isomorphism. We consider the dual of exact sequence (3.1):

$$u^*_{MHS} : 0 \to \mathbf{Z} \to M^*_{\mathbf{Z}} \overset{\pi'}{\to} \mathbf{Z}(-1) \to 0$$

Let γ be the element in $M^*_{\mathbf{Z}}$ such that $\pi'(\gamma) = 2\pi i$. Put $ch(u_{MHS}) = \exp(\langle \omega, \gamma \rangle) \in \mathbf{C}^\times$. Then it is easy to see that $ch(u_{MHS})$ does not depend on the choice of γ.

Let $CH^1(\mathbf{Q}, 1)$ be the Bloch's higher Chow group. (See [6].) In [7], Levine defined an isomorphism $cl : CH^1(\mathbf{Q}, 1) \otimes \mathbf{Q} \to Hom_{DTM_{\mathbf{Q}}}(\mathbf{Q}, \mathbf{Q}(1)[1])$, which is called the cycle map. By using the isomorphism

$$\mathbf{Q}^\times \simeq CH^1(\mathbf{Q}, 1), \tag{3.2}$$

we have an isomorphism $\mathbf{Q}^\times \otimes \mathbf{Q} \to Hom_{DTM_{\mathbf{Q}}}(\mathbf{Q}, \mathbf{Q}(1)[1])$, which is also denoted by cl. We have the following proposition

Proposition 3.3. *The following diagram commutes.*

$$\begin{array}{ccccc}
Hom_{DTM_{\mathbf{Q}}}(\mathbf{Q}, \mathbf{Q}(1)[1]) & \underset{\simeq}{\overset{cl}{\leftarrow}} & \mathbf{Q}^\times \otimes \mathbf{Q} & & \\
\mathbf{H}\downarrow & & \downarrow & & \textit{natural inclusion} \\
Ext^1_{MHS}(\mathbf{Q}, \mathbf{Q}(1)) & \underset{\simeq}{\to} & \mathbf{C}^\times \otimes \mathbf{Q} & &
\end{array}$$

We recall the definition of the cycle map. (See [7].) Let Δ^p be the variety over $\mathbf{Q}$ defined by

$$\Delta^p = \{(x_0, \dots, x_p) \mid \sum_{i=0}^{p} x_i = 1\}.$$

For a subset S of $[0, p]$, the subvariety of Δ^p defined by $x_i = 0$ for $i \in S$ is denoted by δ_S. Let $z^q(\mathbf{Q}, p)$ be the free $\mathbf{Z}$ module generated by codimension q cycles γ of Δ^p, where the codimension of $\gamma \cap \delta_S$ is at least q. Since $\{\Delta^p\}$ is a cosimplicial scheme, we have a complex $z^q(\mathbf{Q}, *)$ using restrictions to faces. The homology of $H_p(z^q(\mathbf{Q}, *))$ is denoted by $CH^q(p)$ and it is called the Bloch's higher Chow group. An element in $CH^1(1) \otimes \mathbf{Q}$ is represented by a $\mathbf{Q}$-linear combination of 0-dimensional sub-schemes $z_1, \dots, z_j$ in Δ^1 defined over $\mathbf{Q}$ which does not intersect with $\delta_{\{0\}} \cup \delta_{\{1\}}$. Equality (3.2) is obtained by attaching the class of z_i to $Nm(z_i - 1)/Nm(z_i) \in \mathbf{Q}^\times$. By using trace, z_i defines a morphism $[z_i] : \mathbf{Q} \to \mathbf{Q}_{\Delta^1}(1)[2]$. After [7], we define $\mathbf{Q}^{\leq 1}_{\Delta^*}(1)[2]$ by

$$\mathbf{Q}^{\leq 1}_{\Delta^*}(1)[2] = (\mathbf{Q}_{\Delta^1}(1)[2]_{\text{degree}=-1} \to \mathbf{Q}_{\Delta^0}(1)[2]^{\oplus 2}_{\text{degree}=0}).$$

Then $\mathbf{Q}^{\leq 1}_{\Delta^*}(1)[2]$ is quasi-isomorphic to $\mathbf{Q}(1)[2]$. The morphism $[z_i]$: $\mathbf{Q}_{\text{degree}=-1} \to \mathbf{Q}_{\Delta^1}(1)[2]_{\text{degree}=-1}$ defines a morphism $\mathbf{Q}[1] \to \mathbf{Q}^{\leq 1}_{\Delta^*}(1)[2]$ and this defines a morphism $cl(z_i) \in Hom_{DTM}(\mathbf{Q}, \mathbf{Q}(1)[1])$. We compute $ch(\mathbf{H}(cl(z_i)))$, where $\mathbf{H}(cl(z_i))$ is the Hodge realization of $cl(z_i)$. For simplicity, we assume $z \in \Delta^1 - \{0,1\}$ is a $\mathbf{Q}$ rational section. Let j be the inclusion $\Delta^1 - \{0,1\} \to \Delta^1$. Then z induces a morphism of mixed Hodge complex:

$$[z] : \mathbf{Q} \to \mathbf{R}\Gamma(\Delta^1, j_!\mathbf{Q}(1)_{\Delta^1 - \{0,1\}})[2] \simeq \mathbf{Q}(1)[1].$$

Then we have the triangle

$$\begin{aligned}\mathbf{Q}(1)[1] \simeq \mathbf{R}\Gamma(\Delta^1, j_!\mathbf{Q}(1)_{\Delta^1 - \{0,1\}}[2]) \to \mathbf{R}\Gamma(\Delta^1 - \{z\}, j_!\mathbf{Q}(1)_{\Delta^1 - \{0,1\}})[2] \\ \to \mathbf{Q}[1]\end{aligned}$$

By taking H^{-1} of complices, we have the exact sequence of mixed Hodge structures:

(3.3)

$$0 \to H^1(\Delta^1, j_!\mathbf{Q}(1)_{\Delta^1 - \{0,1\}}) \to H^1(\Delta^1 - \{z\}, j_!\mathbf{Q}(1)_{\Delta^1 - \{0,1\}}) \xrightarrow{\pi} \mathbf{Q} \to 0.$$

This exact sequence corresponds to the Hodge realization of $cl(z)$. An element $\frac{1}{2\pi i}\frac{dx}{x-z}$ in $H^1_{DR}(\Delta^1 - \{z\}, j_!\mathbf{Q}(1)_{\Delta^1 - \{0,1\}})$ corresponds to $1 \in \mathbf{Q}$ via the projection π in (3.3). The cycle $2\pi i \cdot [0,1]$ defines an element $H_1(\Delta^1 - \{z\}, \mod \{0,1\})(1)$, whose image in $H_1(\Delta^1, \mod \{0,1\})(1)$ is $2\pi i$. Then we have

$$\begin{aligned} ch(\mathbf{H}(cl(z))) &= \exp(2\pi i \int_0^1 \frac{1}{2\pi i}\frac{dx}{x-z}) \\ &= \frac{z-1}{z} \end{aligned}$$

Therefore the diagram in the proposition commutes.

3.3. **Splitting in level** (a_i)**.** Let A be an abelian object in $\mathcal{A}_{TM}$ and $S = \{a_1, \dots, a_k\}$ be a set of integers such that $a_i < a_{i+1}$. An abelian object A in A_{TM} is said to be of type S if $Gr^W_a A = 0$ for $a \notin S$ and the full subcategory of abelian objects in $\mathcal{A}_{TM}$ of type S is denoted by $\mathcal{A}_S$. Note that the category $\mathcal{A}_S$ is stable under taking direct sums and subquotients. Let A be an object of type S and $a = a_i, b = a_{i+1} \in S$. We have the following exact sequence in $\mathcal{A}_{TM}$:

$$0 \to W_aA/W_{a-1}A \to W_bA/W_{a-1}A \to W_bA/W_{b-1}A \to 0.$$

If the morphism $\phi : Hom_{DTM_{\mathbf{Q}}}(W_bA/W_{b-1}A, W_aA/W_{a-1}A[1])$ corresponding to the above exact sequence is zero, A is said to split in level (a,b). If A splits in level (a,b) then a subquotient of A splits in level (a,b). It is easy to see that a objects of type $S - \{a_i\}$ and $S - \{a_{i+1}\}$ split in level (a,b).

Proposition 3.4. *Let A be an object of type S which splits in level (a_i, a_{i+1}). Then A is a subquotient of a module $B = B_1 \oplus B_2$, where (1) B_1 (resp. B_2) is of type $S - \{a_{i+1}\}$ (resp. $S - \{a_i\}$), (2)$W_{a_i}B_1$ (resp. $W_{a_{i-1}}B_2$) is isomorphic to the direct sum of copies of $W_{a_i}A$ (resp. $W_{a_{i-1}}A$) and (3)$B_1/W_{a_{i+1}}B_1$*

(resp. $B_1/W_{a_i}B_2$) is isomorphic to the direct sum of copies of $A/W_{a_{i+1}}A$ (resp. $A/W_{a_i}A$). Here we use the notation $W_{a_0}A = 0$ and $W_{a_{k+1}}A = A$.

Since $Hom_{DTM_{\mathbf{Q}}}(X, Y[2]) = 0$ for abelian objects X, Y, the morphism

$$Hom_{DTM_{\mathbf{Q}}}(X_1, Y[1]) \to Hom_{DTM_{\mathbf{Q}}}(X_2, Y[1])$$

induced by an injective morphism $X_2 \to X_1$ is surjective.

We put $a = a_i, b = a_{i+1}$. By the assumption of the proposition, the exact sequence

$$0 \to W_aA/W_{a-1}A \to W_bA/W_{a-1}A \to W_bA/W_{b-1}A \to 0$$

splits and we have $W_bA/W_{a-1}A = W_aA/W_{a-1}A \oplus W_bA/W_{b-1}A$. By pushing forward the exact sequence

$$0 \to W_bA/W_{a-1}A \to A/W_{a-1}A \to A/W_b \to 0$$

by the morphism $W_bA/W_{a-1}A \to W_aA/W_{a-1}WA$, we have the following exact sequence.

$$0 \to W_aA/W_{a-1}A \to C_1 \to A/W_b \to 0$$

It is easy to see that the morphism $i : A/W_{a-1} \to C_1 \oplus A/W_{b-1}$ is an injective morphism. Therefore one can find an extension

$$0 \to W_{a-1}A \to C_2 \to C_1 \oplus A/W_{b-1} \to 0 \tag{3.4}$$

such that the pull back by the morphism i is isomorphic to

$$0 \to W_{a-1}A \to A \to A/W_{a-1} \to 0.$$

Then $A \to C_2$ is injective. By pulling back exact sequence (3.4) by morphisms $C_1 \to C_1 \oplus A/W_{b-1}$ and $A/W_{b-1} \to C_1 \oplus A/W_{b-1}$, we have the following exact sequences.

$$0 \to W_{a-1}A \to B_1 \to C_1 \to 0,$$
$$0 \to W_{a-1}A \to B_2 \to A/W_{b-1} \to 0.$$

Then the morphism $B_1 \oplus B_2 \to C_2$ is surjective. By pulling back exact sequence (3.4) by the composite morphism $W_aA/W_{a-1}A \to A/W_{a-1} \to C_1 \oplus A/W_{b-1}$, We have an exact sequence

$$0 \to W_aA \to B_1 \to A/W_bA \to 0.$$

Thus we have the proposition.

3.4. **Generators of $\mathcal{A}_{TM}$.** In this section we denote $Hom_{\mathcal{A}_{TM}}(A, B[1])$ by $Ext^1(A, B)$. We define a subset $\mathcal{N}_n$ by

$$\mathcal{N}_n = \{\{a_1, \dots, a_n\} \mid 0 \le a_1 < \cdots < a_n \le n \text{ and } a_{i+1} - a_i \text{ is odd}$$
$$\text{and greater than } 1\}.$$

For $S \in \mathcal{N}_n$, we inductively define a series of objects $\{M_S\}_{S \subset \mathcal{N}}$, where $M_S \in \mathcal{A}_S$ and morphisms $M_S \to \mathbf{Q}(a_n)$ as follows. We set $T = \{a_1, \dots, a_{n-1}\}$.

1. Let $M_{\{a_1\}} = \mathbf{Q}(a_1)$.

2. M_S is defined by the extension

$$0 \to M_T \to M_S \to \mathbf{Q}(a_n) \to 0$$

corresponding to $u \in Ext^1(\mathbf{Q}(a_n), M_T)$ such that $\pi(u) \neq 0$ where π is the natural map

$$Ext^1(\mathbf{Q}(a_n), M_T) \to Ext^1(\mathbf{Q}(a_n), \mathbf{Q}(a_{n-1}))$$

induced by $M_T \to \mathbf{Q}(a_{n-1})$.

By the construction, $Gr_{a_i} M_S \simeq \mathbf{Q}(a_i)$ for $i \in S$. Let $M \in \mathcal{A}_S$. We define $< M, \mathcal{A}_{T_i} >_{T_i \subsetneq S}$ as the minimal sucategroy of $\mathcal{A}_S$ containing $\mathcal{A}_{T_i}$ ($T_i \subsetneq S$) and M and stable under taking direct sums and subquotients.

Proposition 3.5. *For $S \in \mathcal{N}_n$, we have $\mathcal{A}_S =< M_S, \mathcal{A}_{T_i} >_{T_i \subsetneq S}$.*

We omit the proof of the next lemma.

Lemma 3.6. *Let $u_1, u_2 \in Ext^1(A, B)$ corresponding to the following extensions:*

$$0 \to B \to M_1 \to A \to 0,$$
$$0 \to B \to M_2 \to A \to 0.$$

Let M_3 be defined by the extension

$$0 \to B \to M_3 \to A \to 0$$

corresponding to the element $u_1 + u_2 \in Ext^1(A, B)$. Then M_3 is a subquotient of $M_1 \oplus M_2$.

Proof of Proposition 3.5. We prove the proposition by the induction on S. Let M be an element of $\mathcal{A}_S$. We may assume that $Gr_{a_n} M \simeq \mathbf{Q}(a_n)$. Let $S = \{a_1, \dots, a_n\}$ ($a_1 < \cdots < a_n$) and $T = \{a_1, \dots, a_{n-1}\}$. Then $L = W_{a_n - 1} M \in \mathcal{A}_T$. By the hypothesis of the induction, L can be obtained by the subquotient of $M_T^{\oplus m} \oplus \oplus_i L_i$, where $L_i \in \mathcal{A}_{U_i}$ and $U_i \subsetneq T$. That is, there exists an object $N \in \mathcal{A}_T$ and surjective and injective morphisms $M_T^{\oplus m} \oplus \oplus_i L_i \to N$ and $L \to N$. Since $Ext^1(\mathbf{Q}(a_n), M_T^{\oplus m} \oplus \oplus_i L_i) \to Ext^1(\mathbf{Q}(a_n), N)$ is surjective, the object M is a subquotient of the object M_1 defined by the exact sequence

$$0 \to M_T^{\oplus m} \oplus \oplus_i L_i \to M_1 \to \mathbf{Q}(a_n) \to 0.$$

Suppose that M_1 is an object corresponding to $v \in Ext^1(\mathbf{Q}(a_n), M_T^{\oplus m} \oplus \oplus_i L_i)$. Let us write $v = (v_{11}, \dots, v_{1m}, v_{21}, \dots, v_{2k})$, where $v_{1i} \in Ext^1(\mathbf{Q}(a_n), M_T)$ and $v_{2j} \in Ext^1(\mathbf{Q}(a_n), L_j)$. Let $u \in Ext^1(\mathbf{Q}(a_n), M_T)$ be the element corresponding to M_S.

(1) Since $L_i \in \mathcal{A}_{U_i}$ and $U_i \subsetneq T$, the extension A_i corresponding to $v_{2j} \in Ext^1(\mathbf{Q}(a_n), L_j)$ is an element in $\mathcal{A}_{U_i \cup \{a_n\}}$. Note that $U_i \cup \{a_n\} \subsetneq S$.

(2) By the definition of M_S, the image $\pi(u)$ of u is a generator of $Ext^1(\mathbf{Q}(a_n), \mathbf{Q}(a_{n-1}))$. Thus there exist rational numbers $k_1, \dots, k_m$ such that $\pi(v_{1i}) = k_i \pi(u)$. Therefore an extension B_i corresponding to $v_{1i} -$

$k_i u \in Ext^1(\mathbf{Q}(a_n), M_T)$ splits in level (a_{n-1}, a_n). Therefore B_i is a object in $< \mathcal{A}_{T_i} >_{T_i \subsetneq S}$ by Proposition 3.4.

(3) The extension C_i corresponding to $k_i u \in Ext^1(\mathbf{Q}(a_n), M_T)$ is isomorphic to (1) M_S if $k_i \neq 0$, and (2) $M_T \oplus \mathbf{Q}(a_n)$ if $k_i = 0$.

Since M_1 is a subquotient of A_i, B_i, C_i as in (1),(2) and (3), we have the proposition by induction. □

4. Periods of mixed Hodge structures and multiple zeta values

4.1. **Subspace generated by periods.** Let $n \geq 0$ be a natural number and $H = (H_B, H_{DR})$ a $\mathbf{Q}$-Hodge structre over $\mathbf{Q}$ such that $W_{-1}H = 0$ and $W_n H = H$. (Note that W_i denote the $2i$-th weight filtration in the usual convention.) Let $H^* = (H^*_B, H^*_{DR})$ be the dual of H. We define the period space $p_n(H)$ of H of weight n by the $\mathbf{Q}$-linear hull of the set $\{\langle \gamma, \omega \rangle \mid \gamma \in H^*_B, \omega \in F^n H_{DR}\}$. Then $p_n(H)$ is a finite dimensional $\mathbf{Q}$-subvector space in $\mathbf{C}$. The following properties are straight forward from the definition. Let H_1, H_2 be mixed Tate Hodge strucutres such that $W_{-1}H = 0$ and $W_n H = H$.

1. If H_1 is a subquotient of H_2, we have $p_n(H_1) \subset p_n(H_2)$.
2. If $W_{n-1}H_1 = H_1$, then $p_n(H_1) = 0$.
3. We have $p_{n+k}(H(-k)) = (2\pi i)^k p_n(H)$.

For an abelian object $M \in \mathcal{A}_{TM}$, the period space $p_n(\mathbf{H}(M))$ of the Hodge realization of M is also denoted by $p_n(M)$. Since the differnetial form $\omega \in H_{DR}$ is defined over $\mathbf{R}$, the complex conjugation c for tolopolgical realizations and the complex conjugation for the periods $\langle \gamma, \omega \rangle$ commultes, i.e. $\overline{\langle \gamma, \omega \rangle} = \langle \gamma^c, \omega \rangle$. As a consequence, $p(H)$ is stable under the action of complex conjugation. Moreover, the topological action of complex conjugate c acts on the Betti realization $\mathbf{Q}(k)$ as $(-1)^k$-multiplication.

Lemma 4.1. *Let H be an object of $\mathcal{A}_S$, where $S = \{a_1 < \cdots < a_k\}$. Then the spaces $p_{a_k}(H)$ and $p_{a_k}(H/W_{a_1})$ are stable under the complex action. The action of complex conjugate on the space $p_{a_k}(H)/p_{a_k}(H/W_{a_1})$ coincides with the $(-1)^{a_1}$-multiplicaltion.*

Proof. The space $p_{a_k}(H)$ (resp. $p_{a_k}(H/W_{a_1})$) is generated by the set $\{\langle \gamma, \omega \rangle \mid \omega \in F^{a_k} H_{DR}, \gamma \in H^*\}$ (resp. $\{\langle \gamma, \omega \rangle \mid \omega \in F^{a_k} H_{DR}, \gamma \in W_{-a_2} H^*\}$). For an element $\gamma \in H^*_B$, we have $\gamma - (-1)^{a_1} \gamma^c \in W_{-a_2} H^*_B$. Therefore $\langle \gamma, \omega \rangle - (-1)^{a_1} \overline{\langle \gamma, \omega \rangle} \in p(H/W_{a_1})$. □

Let S be a finite subset of $[0, n]$. Let $p_n(S)$ be the $\mathbf{Q}$-subspace of $\mathbf{C}$ generated by $p_n(H)$ for $H \in \mathcal{A}_S$. Then it is easy to see that for finite subsets S, T such that $T \subset S$, we have $p_n(T) \subset p_n(S)$.

4.2. **Periods and multiple zeta values.** In this section, we construct topological cycles of relative homology group and compute the natural pairing

$$H^n_{DR}(Y^0, j_! \mathbf{Q}) \otimes H^B_n(Y^0, \bmod \mathbf{B}^0) \to \mathbf{C},$$

where $\mathbf{B}^0$ is the normal crossing divisor defined in §2.4 and $j : Y^0 - \mathbf{B}^0 \to Y^0$ be the natural inclusion. For an element γ in $\pi_1(\mathbf{A}^1 - \{0, 1\}, \frac{1}{2})$, we define a

path $\bar{\gamma}$ connecting 0 and 1 by $[\frac{1}{2},1]\circ\gamma\circ[0,\frac{1}{2}]$, where $\circ$ denotes the composite of paths. It is well defined up to homotopy equivalece. We define a continuous map δ_γ from $\Delta_n=\{0<t_1<\cdots<t_n<1\}$ to Y^0 by

$$\delta_\gamma(t_1,\dots,t_n)=(\bar{\gamma}(t_1),\dots,\bar{\gamma}(t_1)).$$

Since the variety $(\mathbf{A}^1-\{0,1\})^n$ can be identified with an open set of Y, the map δ_γ can be lifted to a map $\delta_{\gamma,Y}$ to Y.

Proposition 4.2. *The closure $\overline{Im(\delta_{\gamma,Y})}$ of the image $Im(\delta_{\gamma,Y})$ is contained in Y^0. The boundary $\overline{Im(\delta_{\gamma,Y})}-Im(\delta_{\gamma,Y})$ is contained in $\mathbf{B}^0$.*

Proof. For the first assertion, it is enough to prove that $\overline{Im(\delta_{\gamma,Y})}\cap E_i^{pr}$, $\overline{Im(\delta_{\gamma,Y})}\cap F_i^{pr}$ $(i=0,\dots,n-2)$ and $\overline{Im(\delta_{\gamma,Y})}\cap B_i^{pr}$ $(i=0,\dots,n)$ does not intersect with D_Y.

(1) **Proof of $\overline{Im(\delta_{\gamma,Y})}\cap E_i^{pr}\cap D_Y=\emptyset$.** By the natural map $\pi_i:E_i^{pr}\to Y(x_{n-i+1},\dots,x_n)$, $\pi_i(\overline{Im(\delta_{\gamma,Y})}\cap E_i^{pr})$ does not intersect with $D_Y(x_{n-i+1},\dots,x_n)$ by the inductive hypothesis. For $y\in\pi_i(\overline{Im(\delta_{\gamma,Y})}\cap E_i^{pr})$, $\pi_i^{-1}(y)\cap Im(\delta_{\gamma,Y})\cap E_i^{pr,0}$ is equal to

$$0\le\eta_i\le1\quad(i=1,\dots,n-i-1)$$

with the coordinate of (2.9). Therefore $\overline{Im(\delta_{\gamma,Y})}\cap E_i^{pr}$ does not intersect with D_Y.

(2) **Proof of $\overline{Im(\delta_{\gamma,Y})}\cap F_i^{pr}\cap D_Y=\emptyset$.** The proof is similar.

(3) **Proof of $\overline{Im(\delta_{\gamma,Y})}\cap B_i^{pr}\cap D_Y=\emptyset$.** Since $B_i^{pr}\simeq Y(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ for $i=1,\dots,n$ and $B_0^{pr}\simeq Y(x_2,\dots,x_n)$, $\overline{Im(\delta_{\gamma,Y})}\cap B_i^{pr}$ does not intersect with $D_Y\cap B_i^{pr}$ by the inductive hypothesis.

The second assertion is a direct consequence of the first assertion. $\square$

The relative cycle $\overline{Im(\delta_{\gamma,Y})}$ is denoted by $\bar{\delta}_\gamma$ For an element $\eta=\sum_\gamma a_\gamma\gamma\in\mathbf{Q}[\pi_1(\mathbf{A}^1-\{0,1\},\frac{1}{2})]$, we define an element $\bar{\delta}(\eta)$ by $\sum_\gamma a_\gamma\bar{\delta}_\gamma$ in $H_n(Y^0,\bmod \mathbf{B}^0)$. Then we have a linear map

$$\bar{\delta}:\mathbf{Q}[\pi_1(\mathbf{A}^1-\{0,1\},\frac{1}{2})]\to H_n(Y^0,\bmod \mathbf{B}^0).$$

Since the filtration induced by spectral sequence (2.13) coincides with the weight spectral sequence, $F^n(H^n_{DR}(Y^0,j_!\mathbf{Q}_V))$ is generated by differential form $\omega(1,\dots,n;\epsilon_1,\dots,\epsilon_n)$ such that $\epsilon_i=0,1$ $i=2,\dots,n-1$ and $\epsilon_1=1,\epsilon_n=0$. By the definition of $\bar{\delta}$,

$$(-1)^l\langle\bar{\delta}(1),\omega(1,\dots,n;\epsilon_1,\dots,\epsilon_n)\rangle=\zeta(k_1,\dots,k_l),$$

where $\epsilon_j=1$ for $j=h_0+1,h_1+1,\dots,h_{l-1}+1$ and $\epsilon_j=0$ otherwise. Here we use the correspondence between k_i and h_i in the introduction. Therefore $\zeta(k_1,\dots,k_l)\in p_n(H^n(j_!\mathbf{Q}_V,Y^0))$, and we have the following proposition.

Proposition 4.3. $L_n\subset p_n(H^n(Y^0,j_!\mathbf{Q}_V))\cap\mathbf{R}$.

4.3. **Periods of subquotients of $j_!\mathbf{Q}_V[n]$.** In this subsection, we introduce an inductive structure to compute the periods of subquotients of $H^n(Y^0, j_!\mathbf{Q}_V)$.

We define a divisor $\mathbf{B'}^0 = \cup_{i=1}^n \tilde{B}_i^0$ and j' as the open immersion from $V' = Y^0 - \mathbf{B'}^0$ to Y^0. The complement $\mathbf{B}^0 - \mathbf{B'}^0$ is denoted by V_{n-1} and the open immersion $V_{n-1} \to \tilde{B}_0^0$ is denoted by j_{n-1}. It is easy to see that $j'_!\mathbf{Q}_{V'} \mid_{\tilde{B}_0^0} \simeq j_{n-1,!}\mathbf{Q}_{V_{n-1}}$. Therefore we have the following distinguished triangle

$$j_!\mathbf{Q}_V \to j'_!\mathbf{Q}_{V'} \to j_{n-1,!}\mathbf{Q}_{V_{n-1}} \xrightarrow{+1} . \tag{4.1}$$

$\tilde{B}_0^0$ is homotopy equivalent to $B_0^{pr,0} \simeq Y(x_2, \ldots, x_n)^0$, and V_{n-1} is equal to $\tilde{B}_0^0 - \cup_{i=1}^n \tilde{B}_i^0$.

Proposition 4.4. *The morphism*

$$j_{n-1,!}\mathbf{Q}_{V_{n-1}}[n-1] \to j_!\mathbf{Q}_V[n] \tag{4.2}$$

arising from triangle (4.1) is injective and the image is identified with $W_{n-1}j_!\mathbf{Q}_V[n]$.

Proof. Let $\oplus_{\#S=n+1-*,0\in S\subset[0,S]}\mathbf{Q}_{\tilde{B}_S^0}$ and $\oplus_{\#S=n+1-*,S\subset[1,S]}\mathbf{Q}_{\tilde{B}_S^0}$ be the complices defined similarly to (2.12). Then the triangle

$$j_{n-1,!}\mathbf{Q}_{V_{n-1}}[-1] \to j_!\mathbf{Q}_V \to j'_!\mathbf{Q}_{V'}$$

is isomorphic to

$$\begin{aligned}\oplus_{\#S=n+1-*,S\subset[1,S]}\mathbf{Q}_{\tilde{B}_S^0} &\to \oplus_{\#S=n+1-*,S\subset[0,S]}\mathbf{Q}_{\tilde{B}_S^0}\\ &\to \oplus_{\#S=n+1-*,0\in S\subset[0,S]}\mathbf{Q}_{\tilde{B}_S^0}.\end{aligned}$$

By taking the Hodge realization of this triangle, morphism (4.2) is an injective morphism between abelian objects. □

This proposition gives an inductive structre for relative cohomologies. We claim the comptibility of this inductive structure and the homomorphism $\bar{\delta}$.

Proposition 4.5. *Let $H_n(Y^0,\ mod\ \mathbf{B}^0) \to H_{n-1}(\tilde{B}_0^0,\ mod\ \mathbf{B'}^0 \cap \tilde{B}_0^0)$ be the dual of the Hodge realization of the homomorphism given in Proposition 4.4. Then the following diagram commutes.*

$$\begin{array}{ccc}\mathbf{Q}[\pi_1(\mathbf{A}^1-\{0,1\},\frac{1}{2})] & \to & \mathbf{Q}[\pi_1(\mathbf{A}^1-\{0,1\},\frac{1}{2})]\\ \bar{\delta}\downarrow & & \bar{\delta}\downarrow\\ H_n(Y^0,\ mod\ \mathbf{B}^0) & \to & H_{n-1}(\tilde{B}_0^0,\ mod\ \mathbf{B'}^0\cap\tilde{B}_0^0).\end{array}$$

Moreover, we have

$$H_{n-1}(\tilde{B}_0^0,\ mod\ \mathbf{B'}^0 \cap \tilde{B}_0^0) \simeq H_{n-1}(B_0^{pr,0},\ mod\ \mathbf{B}_{n-1}^0),$$

where $\mathbf{B}_{n-1}^0$ is defined by

$$B_0^{pr,0} \cap (E_0^{pr,0} \cup \cdots \cup E_{n-2}^{pr,0} \cup B_2^{pr,0} \cup \cdots \cup B_{n-1}^{pr,0} \cup \tilde{B}_n^0).$$

We compute the period of subquotients of $H^n(Y^0, j_!\mathbf{Q}_V)$ via formulas for iterated integrals. Let $\omega_1, \dots, \omega_k$ be 1-forms on $[0,1]$. The iterated integral $\int_0^x \omega_1 \cdots \omega_k$ is inductively defined by

$$\int_0^x \omega_1 \cdots \omega_k = \int_0^x [\omega_1(u)(\int_0^u \omega_2 \cdots \omega_k)].$$

For the properties of iterated integrals, see [9]. Let γ be a path from $[0,1]$ to $\mathbf{A}^1 - \{0,1\}$ and $\omega_1, \dots, \omega_k$ be holomorphic 1-form on $\mathbf{A}^1 - \{0,1\}$. The iterated integral along the path γ is defined by

$$\int_\gamma \omega_1 \dots \omega_k = \int_0^1 \gamma^*(\omega_1) \dots \gamma^*(\omega_k).$$

It is known that it depends only on the homotopy quivalece class of γ. For a formal $\mathbf{Q}$-linear combination $\gamma = \sum_i c_i \gamma_i$ of paths from p_0 to p_1, we define $\int_\gamma \omega_1 \cdots \omega_k$ by $\sum_i c_i \int_{\gamma_i} \omega_1 \cdots \omega_k$.

Let γ (resp. δ) be paths connecting b and c (a and b) and $\gamma \cdot \delta$ be the composite of these paths. Then we have the following coproduct formula.

$$\int_{\gamma\cdot\delta} \omega_1 \cdots \omega_k = \int_\gamma \omega_1 \cdots \omega_k + \sum_{i=1}^{k-1} (\int_\gamma \omega_1 \cdots \omega_i) \cdot (\int_\delta \omega_{i+1} \cdots \omega_k) + \int_\delta \omega_1 \cdots \omega_k.$$

Let γ be a path from 1 to 0 such that $\gamma = x$ if $x \in [0,\epsilon)$ or $(1-\epsilon, 1]$ for a sufficiently small ϵ. Then

$$\int_\gamma \frac{dx}{x-e_n} \cdots \frac{dx}{x-e_1} = \lim_{\delta\to 0} \int_{\gamma|[\delta,1-\delta]} \frac{dx}{x-e_n} \cdots \frac{dx}{x-e_1}$$

exists if $e_1 = 1, e_n = 0$. It is easy to see that

$$\int_{[\frac{1}{2},1]\gamma[0,\frac{1}{2}]} \frac{dx}{x-e_n} \cdots \frac{dx}{x-e_1} = \langle \bar{\delta}(\gamma), \omega(1,\dots,n; e_1,\dots,e_n \rangle).$$

Let ρ_0 (resp. ρ_1) be a small loop runinging around 0 (resp. 1) with base point $\frac{1}{2}$. The group ring $\mathbf{Q}[\pi_1(\mathbf{A}^1 - \{0,1\}, \frac{1}{2})]$ is denoted by R and the augumentation ideal $Ker(R \to \mathbf{Q})$ is denoted by I.

Proposition 4.6. *We put $\alpha = [\frac{1}{2}, 1]$, $\beta = [0, \frac{1}{2}]$. Suppose that $e_i = 0, 1$ for $i = 2, \dots, n-1$, $e_1 = 0$ and $e_n = 1$.*

1. *If γ is contained in I^{n+1}, then*

$$\int_{\alpha\gamma\beta} \frac{dx}{x-\epsilon_1} \cdots \frac{dx_n}{x-\epsilon_n} = 0$$

2.

$$\int_{\alpha\cdot(g_1-1)\cdots(g_n-1)\cdot\beta} \frac{dx}{x-\epsilon_1} \cdots \frac{dx_n}{x-\epsilon_n} = \begin{cases} (2\pi i)^n & \textit{if } g_i = \rho_{\epsilon_i} \textit{ for all } i \\ 0 & \textit{otherwise} \end{cases}$$

3. *Let* $g_i \in \{\rho_0, \rho_1\}$, $\epsilon_k = 0, 1$ *and* $\epsilon_1 = 1, \epsilon_n = 0$. *Then we have*

$$\int_{\alpha\cdot(g_1-1)\cdots(g_{n-1}-1)\cdot\beta} \frac{dx}{x-\epsilon_1} \cdots \frac{dx_n}{x-\epsilon_n} \in \frac{(2\pi i)^n}{2}\mathbf{Z}.$$

Proof. 1. This equality comes from the coproduct formula.

2. By the coproduct formula, we have

$$\int_{\alpha\cdot(g_1-1)\cdots(g_n-1)\cdot\beta} \omega_1 \cdots \omega_n = \prod_{i=1}^{n} \int_{g_i} \omega_i,$$

and the proposition follows from this equality.

3. We use the notation $(\gamma, \epsilon_1 \cdots \epsilon_k) = \int_\gamma \frac{dx}{x-\epsilon_1} \cdots \frac{dx_k}{x-\epsilon_k}$. By the coproduct formula, we have

$$\begin{aligned} &\int_{\alpha\cdot(g_1-1)\cdots(g_{n-1}-1)\cdot\beta} \frac{dx}{x-\epsilon_1} \cdots \frac{dx_n}{x-\epsilon_n} \\ =&(\alpha, \epsilon_1) \prod_{i=1}^{n-1} (g_i, \epsilon_{i+1}) + (\beta, \epsilon_n) \prod_{i=1}^{n-1} (g_i, \epsilon_i) \\ &+ \sum_{k=1}^{n-1} \prod_{i=1}^{k-1} (g_i, \epsilon_i) \cdot (g_k, \epsilon_k\epsilon_{k+1}) \cdot \prod_{i=k+1}^{n-1} (g_i, \epsilon_{i+1}). \end{aligned} \tag{4.3}$$

Let p (resp. q) be the minimal number such that $g_{p+1} \neq \rho_{\epsilon_{p+1}}$ (resp. maximal number such that $g_{q-1} \neq \rho_{\epsilon_q}$). If $p+1 \leq q-1$, i.e. there exists no k such that $q-1 \leq k \leq p+1$, then all the terms in (4.3) vanish. Therefore we have the proposition. We may assume that $q-1 < p+1$. We can easily see that $q > 1$ or $p < n-1$. Suppose that $q > 1$. If $q-1 < k < p+1$, (resp. $q-1 > k$ or $k > p+1$) then $\prod_{i=1}^{k-1}(g_i, \epsilon_i) \cdot (g_k, \epsilon_k\epsilon_{k+1}) \cdot \prod_{i=k+1}^{n-1}(g_i, \epsilon_{i+1})$ is equal to $(2\pi i)^n/2$ (resp. 0).

(Case 1) If $p = n-1$, we have

$$\prod_{i=1}^{q-2} (g_i, \epsilon_i) \cdot (g_{q-1}, \epsilon_{q-1}\epsilon_q) \cdot \prod_{i=q}^{n-1} (g_i, \epsilon_{i+1}) + (\beta, \epsilon_n) \prod_{i=1}^{n-1} (g_i, \epsilon_i) = 0.$$

Therefore the sum (4.3) is contained in $\frac{(2\pi i)^n}{2}\mathbf{Z}$.

(Case 2) If $p < n-1$, we have

$$\begin{aligned} &\prod_{i=1}^{q-2} (g_i, \epsilon_i) \cdot (g_{q-1}, \epsilon_{q-1}\epsilon_q) \cdot \prod_{i=q}^{n-1} (g_i, \epsilon_{i+1}) \\ &+ \prod_{i=1}^{p} (g_i, \epsilon_i) \cdot (g_{p+1}, \epsilon_{p+1}\epsilon_{p+2}) \cdot \prod_{i=p+2}^{n-1} (g_i, \epsilon_{i+1}) = 0. \end{aligned}$$

Again the sum (4.3) is contained in $\frac{(2\pi i)^n}{2}\mathbf{Z}$.

For the case $q = 1$ and $p < n-1$, the proof is similar. ∎

Set $h_i = \rho_i - 1 \in R$.

Corollary 4.7. *The restriction of the morphism $\bar{\delta}$ in Proposition 4.5 to $\mathbf{Q}\cdot 1 \oplus h_0Rh_1$ is surjective and $H_n(Y^0, \bmod \mathbf{B}^0)$ is isomorphic to $\mathbf{Q}\cdot 1 \oplus h_0Rh_1/h_0I^{n-1}h_1$ via this homomorphism. Moreover the weight filtration coincides with that induced from the power of I.*

Proof. We prove the corollary by induction. For $n = 2$, we can prove the proposition directly. By the duality and the first statement of Proposition 4.6, we have $\bar{\delta}(h_0I^{n-1}h_1) = 0$. The following diagram is commutative by Proposition 4.5.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
h_0I^{n-2}h_1/h_0I^{n-1}h_1 \arrow[r, "(1)"] \arrow[d] & W_{-n}H_n(Y^0, \bmod \mathbf{B}^0) \arrow[d] \\
\mathbf{Q}\cdot 1 \oplus h_0Rh_1/h_0I^{n-1}h_1 \arrow[r, "(2)"] \arrow[d] & H_n(Y^0, \bmod \mathbf{B}^0) \arrow[d] \\
\mathbf{Q}\cdot 1 \oplus h_0Rh_1/h_0I^{n-2}h_1 \arrow[r, "(3)"] \arrow[d] & H_{n-1}(B_0^0, \bmod \mathbf{B'}^0 \cap B_0^0) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Here the columns are exact. Homomorphism (3) is an isomorphism by the assumption of induction. Using the duality, homomorphism (1) is an isomorphism by Proposition 4.6.2. □

Theorem 4.8. *Let $H^n = H^n(Y^0, j_!\mathbf{Q}_V)$. Then the exact sequence*

$$\begin{aligned}(4.4)\qquad 0 \to W_{k-1}(H^n)/W_{k-2}(H^n) &\to W_k(H^n)/W_{k-2}(H^n) \\ &\to W_k(H^n)/W_{k-1}(H^n) \to 0\end{aligned}$$

splits as mixed Hodge structures for all k.

Proof. By the inductive structure introduced in Proposition 4.5, we may assume $k = n$. We compute the extension class of (4.4) in $Ext^1(Gr_n^W(H_n), Gr_{n-1}^W(H_n))$ by the recipe in §3.2. By Proposition 4.6.3, the differential form $\frac{1}{(2\pi i)^n}\omega(1,\dots,n;1,\epsilon_2,\dots,\epsilon_{n-1},0)$ with $\epsilon_i = 0,1$ ($i = 2,\dots,n-1$) forms a $\mathbf{Q}$ base of $W_n(H^n)/W_{n-1}(H^n)$ via the isomorphism

$$F^n(H^n\otimes\mathbf{C}) \simeq (W_n(H^n)/W_{n-1}(H^n))\otimes\mathbf{C}.$$

On the other hand, $\pi(\bar{\delta}(h_0(g_2-1)\cdots(g_{n-2}-1)h_1))$ form a basis in $H_{n-1}(B_0^{pr,0}, \bmod \mathbf{B}_{n-1}^0)$ under the morphism

$$\pi : H_n(Y^0, \bmod \mathbf{B}^0) \to H_{n-1}(B_0^{pr,0}, \bmod \mathbf{B}_{n-1}^0).$$

Therefore the pairing

$$\begin{aligned}&\langle\bar{\delta}(h_0(g_2-1)\cdots(g_{n-2}-1)h_1), \frac{1}{(2\pi i)^n}\omega(1,\dots,n;1,\epsilon_2,\dots,\epsilon_{n-1},0)\rangle \\ =&\frac{1}{(2\pi i)^n}\int_{h_0(g_2-1)\cdots(g_{n-2}-1)h_1}\frac{dx}{x}\frac{dx}{x-e_{n-1}}\cdots\frac{dx}{x-e_2}\frac{dx}{x-1} \in \frac{1}{2}\mathbf{Z}\end{aligned}$$

and the extension class of (4.4) vanishes.

□

We define $\mathcal{N}_n^{\geq 2}$ by

$$\mathcal{N}_n^{\geq 2} = \{\{a_1, \ldots, a_n\} \mid 0 \leq a_1 < \cdots < a_n \leq n \text{ and } a_{i+1} - a_i \geq 2 \text{ for all } i\}.$$

Corollary 4.9. *Let $H^n = j_!\mathbf{Q}_V[n] \in \mathcal{A}_{TM}$. Exact sequence (4.4) splits in the category $\mathcal{A}_{TM}$. $j_!\mathbf{Q}_V[n]$ is a subquotient of a direct product of objects in $\mathcal{A}_S$ with $S \in \mathcal{N}_n^{\geq 2}$.*

Proof. The first statement is a direct consequence of Theorem 4.8 and Proposition 3.3. We prove the second statement by the induction on n. Since $H^n = j_!\mathbf{Q}_V[n]$ splits in level $(n-1, n)$, by Proposition 3.4, there exist an object B_1 and B_2 in $\mathcal{A}$ such that the following statement holds. (1) H^n is a subquotient of $B_1 \oplus B_2$. (2) B_1 is an object of $\mathcal{A}_{[0,n-2]\cup\{n\}}$. $W_{n-2}B_1$ is isomorphic to the direct sum of copies of $W_{n-2}(H^n)$. (3) $W_{n-1}B_2 = B_2$ is isomorphic to the direct sum of copies of $W_{n-1}(H^n)$. By the assumption of induction, $W_{n-2}(H^n)$ (resp. $W_{n-1}(H^n)$) is a subquotient of a direct sum of objects in $A_S \in \mathcal{A}_S$ with $S \in \mathcal{N}_{n-2}^{\geq 2}$ (resp. $S \in \mathcal{N}_{n-1}^{\geq 2}$). Therefore B_1 and B_2 are subquotients of a direct sum of objects in $A_S \in \mathcal{A}_S$ with $S \in \mathcal{N}_n^{\geq 2}$ and $S \in \mathcal{N}_{n-1}^{\geq 2}$, respectively. □

5. Proof of Main Theorem

We use the same notations $\mathcal{N}_n$ and $\mathcal{N}_n^{\geq 2}$ as in §3.4 and §4.3 respectively.

Lemma 5.1. *Let $S = \{a_1 < \cdots < a_l\}$.*

1. *If $S \in \mathcal{N}_n$, then* $\dim p_n(S)/\sum_{T \subsetneq S} p_n(T) \leq 1$.
2. *If $S \in \mathcal{N}_n^{\geq 2} - \mathcal{N}_n$, then* $\dim p_n(S)/\sum_{T \subsetneq S} p_n(T) = 0$.

Proof. If every $a_{k+1} - a_k$ is an odd number greater than 1, $\mathcal{A}_S = < M_S, \mathcal{T} >_{T \subsetneq S}$. Since $M_S/W_{a_1}M_S \in \mathcal{A}_{S-\{a_1\}}$, the set $\{\langle \gamma, \omega \rangle \mid \omega \in F_nM_S, \gamma \in W_{-a_2}M_S^*\}$ is contained in $p_n(S - \{a_1\})$. Since $\dim W_{-a_1}M_S^*/W_{-a_2}M_S^* = 1$ and $\dim F^n(M_S) \leq 1$, $p_n(S)/\sum_{T \subsetneq S} p_n(T)$ is at most one dimensional.

Suppose that $a_{i+1} - a_i$ is even. Let $A \in \mathcal{A}_S$. Since $Ext^1(\mathbf{Q}(-a_i), \mathbf{Q}(-a_{i+1})) = 0$, A splits in level (a_i, a_{i+1}). Therefore A can be written as a subquotient of a direct sum $B_1 \oplus B_2$ with $B_1 \in \mathcal{A}_{S-\{a_i\}}$ and $B_2 \in \mathcal{A}_{S-\{a_{i+1}\}}$ by Proposition 3.4. Therefore we have the second statement. □

By Corollary 4.9, the space $p_n(H^n(j_!\mathbf{Q}_V))$ is contained in the space $\sum_{S \in \mathcal{N}_n^{\geq 2}} p_n(S)$. Since $\sum_{S \in \mathcal{N}_n^{\geq 2}} p_n(S) = \sum_{S \in \mathcal{N}_n} p_n(S)$ by Lemma 5.1, $p_n(H^n(j_!\mathbf{Q}_V))$ is contained in $\sum_{S \in \mathcal{N}_n} p_n(S)$.

Let $\mathrm{op}(a)$ be the cardinality of the set

$$\{(b_1, b_2, \cdots) \text{ (ordered) } \mid b_i \text{ is an odd integer greater than } 1\ , \sum_i b_i = a\}.$$

Since $\#\{S \in \mathcal{N}_n, S \ni a, n, S \subset [a,n]\} = \mathrm{op}(n-a)$, we have

$$\dim(\sum_{\substack{S\in\mathcal{N}_n \\ S\subset[a,n]}} p_n(S)/\sum_{\substack{S\in\mathcal{N}_n \\ S\subset[a+1,n]}} p_n(S)) \leq \mathrm{op}(n-a)$$

by Lemma 5.1. Moreover the complex conjugate c acts on the space $\sum_{S\in\mathcal{N}_n, S\subset[a,n]} p_n(S)/\sum_{S\in\mathcal{N}_n, S\subset[a+1,n]} p_n(S)$ by the $(-1)^a$-multiplication. By Lemma 4.3, L_n is a subset of $p_n(H^n(j_!\mathbf{Q}_V))$ and invariant under the complex conjugation. We have

$$\begin{aligned}\dim L_n &\leq p_n(H^n(j_!\mathbf{Q}_V)) \cap \mathbf{R} \\ &= \sum_{\substack{0\leq a\leq n \\ a\equiv 0(\bmod 2)}} \dim(\sum_{\substack{S\in\mathcal{N}_n \\ S\subset[a,n]}} p_n(S)/\sum_{\substack{S\in\mathcal{N}_{n-1} \\ S\subset[a+1,n]}} p_n(S)) \\ &\leq \sum_{\substack{0\leq a\leq n \\ a\equiv 0(\bmod 2)}} \mathrm{op}(n-a).\end{aligned}$$

Therefore it is enough to prove the following lemma.

Lemma 5.2. *Let d_n be the number defined in the introduction. Then we have*

$$d_n = \sum_{\substack{0\leq a\leq n \\ a\equiv 0(\bmod 2)}} \mathrm{op}(n-a).$$

Proof. We consider variables $u_3, u_5, \dots$, where the degree of u_i is i. We put $V = \oplus_{i=1}^{\infty}\mathbf{Q}u_{2i+1}$ and $\mathbf{V} = \oplus_{i=0}^{\infty}V^{\otimes i}$ and $U = \oplus_{j=0}^{\infty}\mathbf{V}\cdot w^i$, where w is a variable of degree 2. Then the poincare series of U is equal to

$$\sum_{k=0}^{\infty}[\sum_{j=0}^{\infty}(\sum_{i=1}^{\infty}t^{2i+1})^j]t^{2k} = \frac{1}{1-t^2-t^3}.$$

Therefore the coefficient d_n is equal to $\displaystyle\sum_{\substack{a\geq 0 \\ a\equiv 0(\bmod 2)}} \mathrm{op}(n-a)$. □

References

[1] Zagier,D, Values of zeta functions and their applications, *Progr. Math.* **120** (1994), 497–512.

[2] Broadhurst,D.J.-Kreimer.D, Association of multiple zeta values with positive knots via Feynman daigrams up to 9 loops. *Physics Letters B* **vol 393 issues 3-4** (1997), 402–412.

[3] Deligne, P., Letter to Broadhurst.

[4] Goncharov, A. G., Multiple ζ-Values Galois Groups, and Geometry of Modular Varieties, *math.AG/0005069*

[5] Goncharov, A. G., Multiple polylogalithms and mixed Tate motives, *math.AG/0103059*

[6] Bloch, S., Algebraic cycles and higher K-theory. *Adv. in Math.* **vol 61, no.3** (1986), 267–304.

[7] Levine, M., Mixed Motives, *Mathematical Surveys and Monographs* **57** (1998)

[8] Levine, M., Tate motives and the vanishing conjectures for algebraic K-theory. *Algegraic K-theory and algebraic Topology, NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci.* **407** (1993), 167–188.

[9] Chen, K. T., Iterated path integrals. *Bull Amer. Math. Soc.* **83** (1977), 323–338.

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